On α^*-I-Open Set and α^*-I-Continuous Map

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Abstract - *The open sets in topological space are $\alpha - I -$open sets and also they are g-open sets. By considering the idea of Ideal and g-open sets, the new class of open sets ($\alpha^* - I -$open sets) define by generalization of g-open sets. In this paper, we define the $\alpha^* - I -$open setsand establish continuous map in ideal topological spaces. Also, westudy some basic properties of $\alpha^* - I -$open sets.*

Keywords: *Ideal topological spaces, g-open set, g-closed.*

I. INTRODUCTION

In 1965 O. Njastad [1] introduced the concept of α-open set in topological space. The concept of generalized closed sets introduced by N. Levine [2] in 1970. $\alpha^* -$open set has been defined by Missier and Rodrigo [3] in 2013. The notion of ideal in topological spaces introduced by Krotowski [4] and Vaidyanathswamy[5]. The generalization of some important properties in general topology via ideal initiated by Jankovic and Hemlet [6] in 1990.

II. PRELIMINARIES

In this article(X,τ) is a topological space and $A \subseteq X$, $cl(A)$& $i\ (A)$ denote the closure of A and interior of A respectively.

An ideal I is a non empty collection of subsets of set X which is satisfies the conditions:**(a)**$A \in I$ and $B \subset A$ then $B \in I$, **(b)**$A, B \in I$ then $A \cup B \in I$.

The topological space(X,τ) with ideal I on X,i.e.(X,τ,I) is anideal topological space [6]. Given an ideal topological space(X,τ,I). A set operator $(\ \ )^\star: \wp(X) \to \wp(X)$ isa Local function $A^\star$ of $A \subseteq X$ with respect to I and $\tau$is defined as$A^\star = \{x \in X: A \cap U \in I$ for each neighbourhood U of $x\}$.It is also denoted by $A^\star(I)$.

The ideal topology$\tau^\star$ is defined as $\tau^\star = \{X - Cl^\star(A): Cl^\star(A) = A\,, A \subset X\}$, where $Cl^\star(A) = A \cup A^\star$ is a topology via ideal with respect to topology on X. We denote ideal topological space (X,τ,I) by $(X,\tau^\star)$, where$\tau^\star$ is an ideal topology generated via ideal I with respect to topology τ on X. And $\mathcal{B} = \{U - I : U \in \tau, I \in \mathcal{I}\}$ is a basis for topology $\tau^\star$, where $\tau^\star$ is the topology on X obtained from the Kuratowski closure operator $Cl^\star: \wp(X) \to \wp(X)$, defined as $Cl^\star(A) = A \cup A^\star, A \subset X$.

Definition 1:A subset A of a space (X,τ) is said to be ***g-closed*** [2] if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

Definition 2:A subset A of a space (X,τ) is said to be ***g-open*** [2] if $X - A$ is g-closed in X.

Remark 1: Every open set is g-open set but converse is not true.

Definition 3:Let (X,τ) is a topological space and $A \subseteq X$, then the ***g-closure*** [7] of A is defined as the intersection of all g-closed sets in X containing A and is denoted by $Cl*(A)$.

Definition 4:Let (X,τ) is a topological space and $A \subseteq X$, then the***g- interior*** [7] of $A$is defined as the union of all g-open sets in X that are contained in A and is denoted by $i*(A)$.

Definition 5:A subset A of a space (X,τ) is said to be α-**open**[1] if $A \subseteq i(Cl(i(A)))$.

Definition 6:A subset A of a space (X,τ) is called **α* - open**[3] if $A \subseteq i*(\ Cl(\ i*\ (\ A\ )))$.

Definition 7:A subset A of an ideal space (X,τ,I) is said to be $\boldsymbol{\alpha - I -}$***open*** [1] if $A \subseteq i(Cl^\star(i(A)))$.

Theorem 1: (cf. [6]) Let (X,τ,I) be an ideal topological space and $A \subseteq X$, then $Cl^\star(A) \subset Cl(A)$.

Theorem 2:(cf. [6])Let (X,τ,I) be an ideal topological space and $A \subset X$. If $I = \{\emptyset\}$, then $Cl^\star(A) = Cl(A)$.

III. α* -*I*-OPEN

Definition 8:A subset A of an ideal space (X,τ,I) is said to be ***α*-I-open*** if $A \subseteq i*(Cl^\star(i*(A)))$.

Theorem 3: In ideal topological space, eachg-open set is α* -*I-open.*

Proof: Suppose A be ag-open set in X. i.e.$A = i*(A)$. Then $A = i*(A) \subseteq i*\big(Cl^\star(A)\big) = i*(Cl^\star(i*(A)))$. Hence A is α* -I-open

Remark 2:The converse of above results is not necessarily true, we have following example.

Example 1:Let $X = \{a, b, c, d\}$ be the ideal topological space with respect to topology$\tau = \{\phi, \{b\}, \{b, c, d\}, X\}$ and ideal $I = \{\phi, \{a\}\}$ on X. Consider the set $A = \{a, b\}$in X is α*-I-open set but not an open set.

Theorem 4: In ideal topological space, each open set is α^* *-I-open.*

Proof: Suppose A be an open set in X. Since every open set g-open i.e. A is a g-open set. So by the theorem 3.1, set A of X is an α* -I-open. Hence each open set is an α* -I-open.

The converse of above results is not necessarily true, we have following example.

Example2: Let $X = \{a, b, c, d\}$ be the ideal topological space with respect to topology $\tau = \{\phi, \{b\}, \{b, c, d\}, X\}$ and ideal $I = \{\phi, \{a\}\}$ on X. Consider a set $A = \{a, b\}$ is α*-I-open set but not an open set.

Proposition1: Let (X, τ, I) be an ideal topological space and A be non – empty α*-I-open set in X. Then i*(A) is non-empty.

Proof: Let X be an ideal topological space and A be a non-empty α*-I-open set in X. Then by the definition of α*-I-open set, we have $A \subseteq i^*(Cl^\star(i^*(A)))$. Since $A = \phi$ we find that $i^*(Cl^\star(i^*(A))) = \phi$. This implies $i^* \neq \phi$. Hence non-empty α* -I-open has non-empty g-interior.

Theorem 5: Let (X, τ, I) be an ideal topological space and $A \subseteq X$. Then A is α* -I-open iff there exists a g-open set U in X such that $U \subseteq A \subseteq i^*(Cl^\star(U))$.

Proof: Let X be an ideal topological space and A be an α* -I-open in X. This means $A \subseteq i^*(Cl^\star(i^*(A)))$. Put $U = i^*(A)$. Then U is a g-open set in X and we have, $A \subseteq i^*(Cl^\star(U))$. As $U = i^*(A) \subset A$, we find that $U \subseteq A \subseteq i^*(Cl^\star(U))$.

Conversely, let U be a g-open set in X such that $U \subseteq A \subseteq i^*(Cl^\star(U))$. *Now* $U \subset A$, *show that* $U = i^*(U) \subset i^*(A)$, *which implies* $i^*(Cl^\star(U)) \subset i^*(Cl^\star(i^*(A)))$. *As,* $A \subseteq i^*(Cl^\star(U))$, *we find that* $A \subseteq i^*(Cl^\star(i^*(A)))$. Thus A is an α^*-I-open set in X.

Proposition2: Let X be an ideal topological space and $\{A_k\}_{k\in\wedge}$ be the family of α^*-I –open sets in X. Then $\cup_{k\in\wedge} A_k$ is α^*-I –open.

Proof: Let X be an ideal topological space and $\{A_k\}_{k\in\wedge}$ be the family of α^*-I –open sets in X. This means, $A_k \subseteq i^*(Cl^\star(i^*(A_k)))\ \forall\ k \in \wedge$. Put $A = \cup_{k\in\wedge} A_k$. Then we have, $i^*(Cl^\star(i^*(A))) = i^*(Cl^\star(i^*(\cup_{k\in\wedge} A_k))) \supset i^*(Cl^\star(\cup_{k\in\wedge}(i^*(A_k)))) \supset i^*(\cup_{k\in\wedge}(Cl^\star(i^*(A_k)))) \supset \cup_{k\in\wedge}(i^*(Cl^\star(i^*(A_k)))) \supset \cup_{k\in\wedge}(A_k) = A$. i.e. $A \subseteq i^*(Cl^\star(i^*(A)))$. Hence $\cup_{k\in\wedge} A_k$ is α^*-I –open.

Remark 3: *Intersection of two* α^**-I-open set need not be* α^**-I-open set, we have following example.*

Example3: Let $X = \{a, b, c, d\}$ be the ideal topological space with respect to topology $\tau = \{\phi, \{a, b\}, \{a, b, c\}, X\}$ and ideal $I = \{\phi, \{d\}\}$ on. Consider a set $A = \{a, c, d\}$ and $\{b, d\}$ are α*-I-open set but intersection of these set is $\{d\}$ is not α*-I-open set.

IV. VARIOUS TYPE OF SETS IN THE TOPOLOGICAL AND IDEAL TOPOLOGICAL SPACE

In this section, we have studied the relationship between various type of generalised open sets in topological and ideal topological spaces. We begin with relationship between α^* -I-open, α-I-open, α^*-open and α-open in ideal topological spaces.

Theorem 6: In ideal topological space, every α *-I-open is* α^**-I-open.*

Proof: Let A be a non-empty α *-I-open* set in X. Then, $A \subseteq i\left(Cl^\star(i(A))\right) \subseteq i^*(Cl^\star(i^*(A)))$. Hence A is α^*-I-open.

The converse of above results is not necessarily true, we have following example.

Example 4: Let $X = \{a, b, c, d\}$ be the ideal topological space with respect to topology $\tau = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, X\}$ and ideal $I = \{\phi, \{a\}\}$ on X. Consider a set $A = \{b\}$ is α*-I-open set but not α-I-open set.

Remark 4: *The concept of* α^**-I-open set and* α*-open set are independent of each other.*

Example 5: An α*-I-open set may not α -open set.

Let $X = \{a, b, c, d\}$ be the ideal topological space with respect to topology $\tau = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, X\}$ and ideal $I = \{\phi, \{a\}\}$ on X. Consider a set $A = \{b, c\}$ is α*-I-open set but not α-open set.

Example 6: An α-open set may not α*-I-open set.

Let $X = \{a, b, c, d\}$ be the ideal topological space with respect to topology $\tau = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, X\}$ and ideal $I = \{\phi, \{a\}\}$ on X. Consider a set $A = \{a, c, d\}$ is α-open set but not α*-I-open set.

Example 7: A set neither an α-open set nor α*-I-open set.

Let $X = \{a, b, c, d\}$ be the ideal topological space with respect to topology $\tau = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, X\}$ and ideal $I = \{\phi, \{a\}\}$ on X. Consider a set $A = \{c, d\}$ is neither an α-open set nor α*-I-open set.

Theorem 7: In ideal topological space, every α^* *-I-open is* α^**-open.*

Proof: Let A be a non-empty open set in X. Every open set is α^* *-I-open*. Then, $A \subseteq i^*(Cl^\star(i^*(A)))$. Clearly, $A \subseteq i^*(Cl(i^*(A)))$. Hence A is α^**-open.*

Example 8: Converse is not true

Let $X = \{a, b, c, d\}$ be the ideal topological space with respect to topology $\tau = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, X\}$ and ideal $I = \{\phi, \{a\}\}$ on X. Consider a set $A = \{a, c, d\}$ is α*-open set but not α*-I-open set.

Remark 5: *From the above result and example, we have following diagram.*

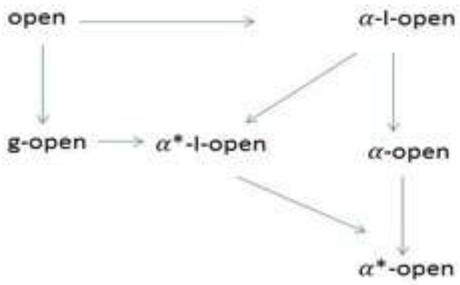

Fig. 1 Relations between different open sets

V. α*-I-CONTINUOUS MAPS

Definition9: A function $f: (X, \tau, I) \to (Y, \sigma)$ is α*-I-continuous map if $f^{-1}(V)$ is a α*-I-open of (X, τ, I) for every open set V of (Y, σ).

Theorem 8: Every continuous maps is α*-I-continuous map.

Proof:Supposef: $(X, \tau, I) \to (Y, \sigma)$ is a map. Let V be an open set of (Y, σ). Since f is continuous map $f^{-1}(V)$ is open in X. Clearly, $f^{-1}(V)$ is a α*-I-open in (X, τ, I).Hence f is an α*-I-continuous map.

Remark 6: *Converse is not true.*

Example9: Let $X = \{a, b, c, d\}$ be an ideal topological space with respect to topology $\tau = \{\phi, \{a\}, \{b, c\}, \{a, b, c\}, X\}$, ideal $I = \{\phi, \{b\}\}$ on X and $Y = \{a, b, c, d\}$ be another topological space with respect to topology $\sigma = \{\phi, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, Y\}$ on Y. Let $f: (X, \tau, I) \longrightarrow (Y, \sigma)$ be defined by f(a) =a, f(b) = b, f(c) = c, f(d) = d. Clearly, f is *an* α*-I-*continuous*. But $f^{-1}(\{b\}) = \{b\}$ is not open in X. Therefore f is not continuous.

Theorem 9:Every α-I-continuous maps is α*-I-continuous map.

Proof:Suppose $f: (X, \tau, I) \to (Y, \sigma, J)$ is a map. Let V be an open set of (Y, σ, J). Since f is continuous map $f^{-1}(V)$ is α-I-open in X. By the theorem 4.1,$f^{-1}(V)$ is a α*-I-open in (X, τ, I).Hence f is an α*-I-continuous map.

Remark 7: *Converse is not true.*

Example 10: *Let* $X = \{a, b, c, d\}$ *be the ideal topological space with respect to topology*$\tau = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, X\}$ *and ideal* $I = \{\phi, \{a\}\}$ *on* X*. And* $Y = \{a, b, c, d\}$ be another topological space with respect to topology $\sigma = \{\phi, \{a\}, \{a, b\}, Y\}$ on Y. Let $f: (X, \tau, I) \longrightarrow (Y, \sigma)$ be defined by f(a) =b, f(b) = a, f(c) = c, f(d) = d. Clearly, f is *an* α*-I-*continuous*. But $f^{-1}(\{a\}) = \{b\}$ is not α-I-open set in X. Therefore f is not α-I-continuous.

***Theorem10*:** Every g-continuous maps is α*-I-continuous map.

Proof: Suppose $f: (X, \tau, I) \to (Y, \sigma)$ is a map. Let V be an open set of (Y, σ). Since f is continuous map $f^{-1}(V)$ is g-open in X. Clearly, $f^{-1}(V)$ is a α*-I-open in (X, τ, I).Hence f is anα*-I-continuous map.

Remark 8: *Converse is not true.*

Example11:Let $X = \{a, b, c, d\}$ be the ideal topological space with respect to topology$\tau = \{\phi, \{a, b\}, \{a, b, c\}, X\}$ and ideal $I = \{\phi, \{a\}\}$ on X. And $Y = \{y_1, y_2, y_3, y_4\}$ be another topological space with respect to topology $\sigma = \{\phi, \{y_2, y_3\}, Y\}$ on Y. Let $f: (X, \tau, I) \longrightarrow (Y, \sigma)$ be defined by $f(a) = y_1$,$f(b) = y_2$, $f(c) = y_4$, $f(d) = y_3$. Clearly, f is an α*-I-continuous. But $f^{-1}(\{y_2, y_3\}) = \{b, d\}$ is not g-open set in X. Therefore f is not g-continuous map.

Theorem12: Let f is anα*-I-continuous map and g is continuous map then $g \circ f$ is an α*-I-continuous map.

Proof:Suppose $f: (X, \tau, I) \to (Y, \sigma)$ is a map. Let V be an open set of (Y, σ). Since f is continuous map $f^{-1}(V)$ is g-open in X. Clearly,$f^{-1}(V)$ is anα*-I-open in (X, τ, I).Hence f is an α*-I-continuous map.

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