Graphs and Codes

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Abstract. *Binary codes from graphs have been studied widely since 1960's. Matrices associated with the graphs were considered as a good tool to construct codes from graphs. Incidence matrix, Adjacency matrix, cut set matrix, circuit matrix etc. were widely used to construct codes with desirable properties. Here we introduce a new binary code –**the vertex code** C from a given graph G, depending on the degree of the vertices of G, in such a way that the vertex polynomial of G is same as the weight enumerator of C.*

Key words: Graph, vertex degree, binary code.

Introduction

Let $G=(V,E)$ be any simple graph of order p and size q. $deg(v_i)$ denotes the degree of vertex v_i, $\delta(G)$ and $\Delta(G)$ represents the minimum and maximum degree of G. For the definitions and algorithms from graph theory we follow [3].

A binary block code C of length n is a collection of binary words of length n. A word is a sequence of binary digits. In a binary word the only digits are 0 and 1. The size of C denoted as $|C|$ is the number of words of C.

If $C=\{c_1,c_2,c_3,\dots,c_n\}$, the Hamming distance between any two code words c_i and c_j is defined as $d_H(c_i,c_j)=Number\ of\ coordinates\ in\ which\ c_i\ and\ c_j\ differ$. The minimum distance of C is the $min\{d_H(c_i,c_k);\ c_i\neq c_{k;}\ \forall(c_i,c_k)\in C\}$. The Hamming weight of a code word c_i is the number of ones in it and is denoted as $wt_H(c_i)$. For coding concepts we refer to [1].

1. Codes from Graphs

A non linear code C can be constructed from certain classes of graphs such that to every vertex u_i, of the given graph G, there corresponds a code word c_i of C satisfying:

i. $deg(u_i)=wt_H(c_i)$.
ii. To every pair of adjacent vertices (u_i,u_j) of G, there corresponds code words c_i, c_j having the property, $d_H(c_i,c_j)=\begin{cases}|deg(u_i)-deg(u_j)|; if\ deg(u_i)\neq deg(u_j)\\ 2, if\ deg(u_i)=deg(u_j)\end{cases}$.
iii. The block length n of C is chosen to be the minimum n satisfying $\binom{n}{j}\geq number\ of\ vertices\ of\ degree\ j\ in\ G;\ \delta(G)\leq j\leq\Delta(G)$.
iv. Thus the size of C is equal to the order of G.

The code so formed gives an idea about the number of vertices and the vertex degrees of the graph G and hence called vertex code of G.

Definition The ***vertex code*** C of a graph G of order p is a collection of p code words satisfying the conditions i.-iii. mentioned above.

The condition ii., is here after termed as distance-degree criteria of code and graph.

Algorithm to check the existence of the vertex code C of a graph G of order p.

Remark Since G is a simple graph of order p, no vertex in G can have degree greater than $(p-1)$, also at least two vertices will have same degree.

Algorithm

1. Write the degree sequence of G and let a_i be the number of vertices with degree i.
2. Fix the block length n of code C to be the minimum n satisfying $\binom{n}{i} \geq a_i$ for every a_i.
3. Start from a vertex v_k with maximum degree, assign a code word c_k arbitrarily to v_k, with length n as on step 2. and $wt_H(c_k) = deg(v_k)$.
4. If more than one vertex has maximum degree, choose the first vertex arbitrarily, and assign a code word with weight equal to its degree. Next vertex is arbitrarily chosen and assigned code word as earlier. Continue this process till all the vertices with maximum degree are assigned code words.
5. Next higher degree vertex is selected and code word of length n and Hamming weight equal to the degree of vertex is assigned to it, satisfying the distance-degree criteria with the code words assigned in steps 3. and 4.
6. A vertex v_j having degree just one less than the maximum degree can be assigned a code word c_j only if v_j is adjacent to at most two vertices say v_k and v_l of maximum degree. Also if there exist any other vertex v_f such that $deg(v_j) = deg(v_f)$ and v_f is also adjacent to v_k and v_l, then no distinct code words c_j and c_f can be assigned to the vertices v_j and v_f. Then go to step 9. and if no such v_f exists 7.
7. Choose the next highest vertex degree and assign code words as in 5.
8. Repeat steps 5. → 7. till we reach 9. or 10.
9. No vertex code of G exists.
10. The code words with respect to all vertices of G makes the vertex code C of G, such that $|C| = p$.

Remark: The vertex code of a graph is not unique.

To find the vertex code of given graphs using the algorithm

Example 1

To show the existence of the vertex code for the graph G in figure 1.

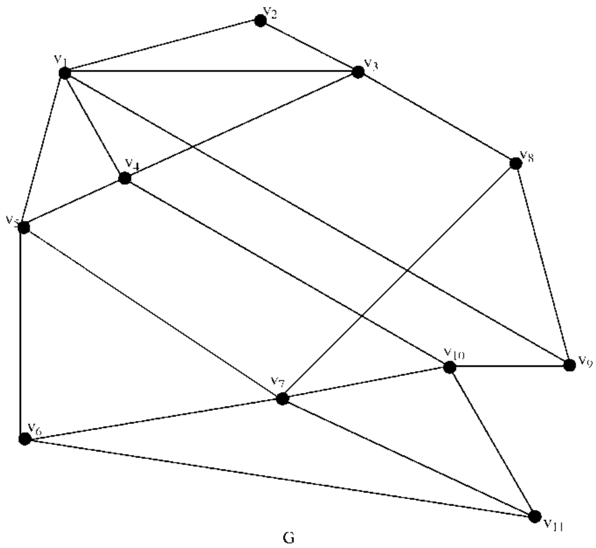


Figure 1

- The degree sequence is $5, 5, 4, 4, 4, 4, 3, 3, 3, 3, 2$; $a_5 = 2$; $a_4 = 4$; $a_3 = 4$; $a_2 = 1$.
- The block length $n = 6$
- The maximum degree ($5°$) vertices v_1 and v_7 are assigned code words c_1 and c_7 of length 6 and Hamming weight 5. Let

$$c_1 = 111110$$
$$c_7 = 111101$$

- The vertices with then the maximum degree$(4°)$ are v_3, v_4, v_5, v_{10} . The vertex v_5 is adjacent to both v_1 and v_7. So it is given a preference over v_3, v_4 and v_{10} and assigned a code word c_5, such that
$$d_H(c_1, c_5) = 1 = d_H(c_7, c_5)$$
Hence, $c_5 = 111100$.
Then code words c_3, c_4 and c_{10} are assigned to vertices v_3, v_4 and v_{10}, such that,
$$d_H(c_1, c_4) = d_H(c_1, c_3) = d_H(c_7, c_{10}) = 1$$
$$d_H(c_4, c_3) = d_H(c_4, c_{10}) = 2$$
Thus,
$$c_3 = 101110$$
$$c_4 = 110110$$
$$c_{10} = 110101$$
- Now the vertices v_6, v_8, v_9, v_{11} have degree 3 each and all have equal priority in getting assigned a code word. So choosing randomly and assigning code words c_6, c_8, c_9, c_{11} respectively to the vertices v_6, v_8, v_9, v_{11} only based on distance - degree criteria of:
$$d_H(c_5, c_6) = 1\ ;\ d_H(c_7, c_6) = 2$$
$$d_H(c_6, c_{11}) = 2\ ;\ d_H(c_7, c_8) = 2$$
$$d_H(c_3, c_8) = 1;\ d_H(c_8, c_9) = 2$$
$$d_H(c_1, c_9) = 2\ ;\ d_H(c_{10}, c_9) = 1$$
$$d_H(c_7, c_{11}) = 2\ ;\ d_H(c_{10}, c_{11}) = 1$$
Thus,
$$c_6 = 111000 \qquad c_8 = 101100$$
$$c_9 = 110100 \qquad c_{11} = 110010$$
- Then the only two degree vertex v_2 is assigned code word c_2 satisfying,
$$d_H(c_1, c_2) = 3$$
$$d_H(c_3, c_2) = 2$$
Thus,
$$c_2 = 101000$$
- Thus the vertex code C of $G_2$is :
$$C = \begin{Bmatrix} 111110, 101000, 101110, 110110, 111100, 111000, \\ 111101, 101100, 110100, 110101, 110001 \end{Bmatrix}$$

Example 2

To show the non existence of the vertex code of graphs G in figure 2 and G_1 in figure 3. (Considered as case 1 and case 2)

Case 1

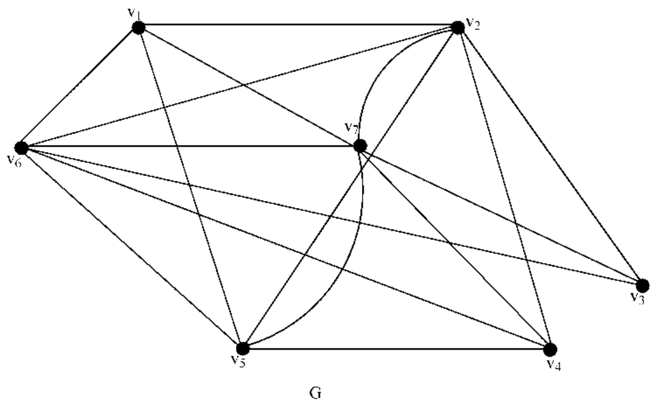


Figure 2

- Following the algorithm the block length n of the required code C can be fixed as $n = 7$.

Here the maximum degree $(6°)$ vertices are v_2, v_6, v_7. The vertex having degree next to the maximum degree is v_5 (of degree 5).

- v_2, v_6, v_7 are assigned code words c_2, c_6, c_7 of length 7 and Hamming weight 6 arbitrarily so that
$$c_2 = 1111110$$

$$c_6 = 1111101$$
$$c_7 = 1111011$$

- Here we cannot find a code word c_5 such that
$$d_H(c_2, c_5) = 1$$
$$d_H(c_6, c_5) = 1$$
$$d_H(c_7, c_5) = 1$$
- Thus G cannot be vertex coded

Case 2

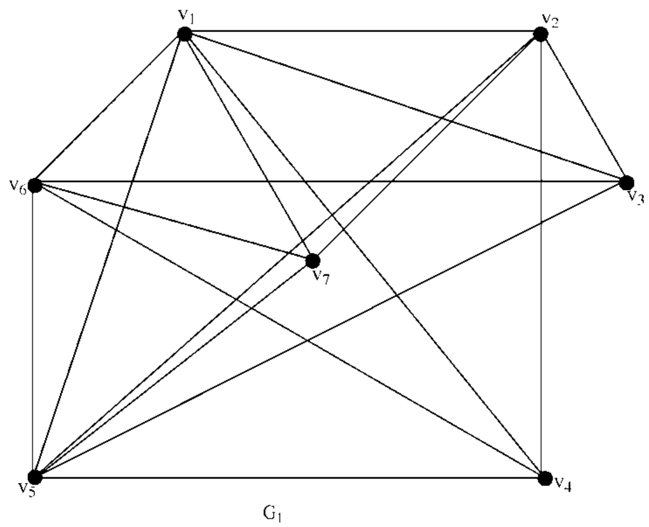


Figure 3

- Degree sequence is $6, 6, 5, 5, 4, 4, 4$. So that $a_6 = 2;\ a_5 = 2;\ a_4 = 3$.
- Block length n is chosen to be the minimum satisfying all $\binom{n}{6} \geq 2;\ \binom{n}{5} \geq 2;\ \binom{n}{4} \geq 3$. Thus $n = 7$.
- The vertices v_1 and v_5 with maximum degree are arbitrarily assigned two code words c_1 and c_5 with length 7 and Hamming weight 6, where
$$c_1 = 1111110$$
$$c_5 = 1111101$$
- The vertices v_2 and v_6 have vertex degrees 5. The code words c_2 and c_6 must be of length 7 and Hamming weight 5, also they should satisfy,

$$d_H(c_1, c_6) = 1 = d_H(c_5, c_6)$$

and

$$d_H(c_1, c_2) = 1 = d_H(c_5, c_2)$$

Since, there exists only one code word 1111100 satisfying these conditions either v_2 or v_6 can be assigned a code word.

- Thus the vertex coding of graph G_1 ends in failure.

Result Let G be a graph having vertex code C. Then the vertex polynomial of G is the same as that of the weight enumerator of C.

We already have for any given graph G, its vertex polynomial [2], $S_G(x) = \sum_{j=0}^{\Delta(G)} a_j x^j$, where a_j is the number of vertices with degree j. Since our study is restricted only to connected graphs, $S_G(x) = \sum_{j=1}^{\Delta(G)} a_j x^j$.

In the vertex code we defined here the code words depend on the vertex degrees of the corresponding graphs. Hence the Hamming weight of each code word equals the degree of the corresponding vertex.

Also in [2] the weight enumerator of a code $C$is defined as $W_C(x) = \sum_{k=1}^{n} c_k x^k$, where c_k denotes the number of code words with Hamming weight k.

In vertex code $\delta(G) \leq k \leq \Delta(G)$.

Thus $W_C(x) = S_G(x)$

$$d_H(c_2, c_4) = d_H(c_3, c_5) = 2$$
$$d_H(c_1, c_5) = d_H(c_3, c_4) = d_H(c_4, c_5) = 4$$

We have the distance of a Code, $d(C) = min\{d_H(c_i, c_j), c_i, c_j \in C, \forall\, c_i \neq c_j\}$.
Thus the vertex code of cycle C_m is of distance two.

2. Graphs from codes

Earlier we constructed vertex code from a graph subject to certain conditions. Here we try to retrace a graph from a given code assuming it represents the vertex code of some graph.

To check whether the given code $C = \{c_1, c_2, c_3, \ldots, c_n\}$ can be retraced to a graph G.

1. If $|C| = n$, it is assumed that the graph G to be constructed from C is of order n.
2. Write the weight enumerator $W_C(x)$ of C, which in turn gives the vertex polynomial of G. Thus giving the number of vertices with particular degrees.
3. Write down the degree sequence.
4. Check whether the degree sequence is graphical.*[3]
5. If the degree sequence is graphical, evaluate, $d_H(c_i, c_j)\ \forall c_i, c_j \in C$.
6. If $d_H(c_i, c_j) = 2$ and $wt_H(c_i) = wt_H(c_j)$, then c_i and c_j may represent the code words of two adjacent vertices v_i and v_j having same degree of the required graph.
7. Find the probability of each code word to be adjacent with the other.
8. Every code word with weight m should be in degree- distance relation with at least m code words, if not no such graph G exists with given C as vertex code.
9. Try to draw the graph by plotting n vertices, the given n code words and join the vertices satisfying degree- distance relation.

To check the existence of graphs and whether it is retrace for a given code

Example 3

Consider the code, $C_1 = \{110010, 100110, 011010, 001110, 000111, 101010\}$.

From the given code C_1, $W_{C_1}(x) = 6x^3 = S_{G_1}(x)$

The degree sequence $3, 3, 3, 3, 3, 3$, is graphical.

$$\Rightarrow the\ expected\ graph\ G_1\ from\ C_1\ is\ 3 - regular\ of\ order\ 6$$

Let the Code C_1 be rewritten as $C_1 ,= \{c_1, c_2, c_3, c_4, c_5, c_6\}$, where c_i is the i^{th} word of C_1.
Now,

$$d_H(c_1, c_2) = 2 \qquad d_H(c_1, c_3) = 2 \qquad d_H(c_1, c_4) = 4$$

$$d_H(c_1, c_5) = 4 \qquad d_H(c_1, c_6) = 2 \qquad d_H(c_2, c_3) = 4$$

$$d_H(c_2, c_4) = 2 \qquad d_H(c_2, c_5) = 2 \qquad d_H(c_2, c_6) = 2$$

$$d_H(c_3, c_4) = 2 \qquad d_H(c_3, c_5) = 4 \qquad d_H(c_3, c_6) = 2$$

$$d_H(c_4, c_5) = 2 \qquad d_H(c_4, c_6) = 2 \qquad d_H(c_5, c_6) = 4$$

Since the weight of each code word is 3, every word should have at least 3 words with distance 2 (satisfying degree- distance criteria). But here the code word $c_5 = 000111$, has distance 2 only with code words $c_2 =$

100110 and $c_4 = 001110$. Hence C_1 cannot be the vertex code with respect to any graph G_1 with vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$.

Example 4

Consider the code $C_2 = \{11111, 11100, 11010, 10110, 01110, 01101\}$

Solution: From the given code C_2, $W_{C_2}(X) = x^5 + 5x^3 = S_{G_2}(x)$.

$$|C_2| = 6 = order\ of\ required\ graph\ G_2$$

The degree sequence 5, 3, 3, 3, 3, 3 is graphical.

Let the code C_2 be re written as $C_2 = \{c_1, c_2, c_3, c_4, c_5, c_6\}$ where,

$$c_1 = 11111 \quad c_2 = 11100 \quad c_3 = 11010$$

$$c_4 = 10110 \quad c_5 = 01110 \quad c_6 = 01101$$

We have $d_H(c_1, c_j) = 2, \forall j = 2,3, \ldots, 6$, and

$$d_H(c_2, c_3) = 2 \quad d_H(c_2, c_4) = 2 \quad d_H(c_2, c_5) = 2 \quad d_H(c_2, c_6) = 2$$

$$d_H(c_3, c_4) = 2 \quad d_H(c_3, c_5) = 2 \quad d_H(c_3, c_6) = 4 \quad d_H(c_4, c_5) = 2$$

$$d_H(c_4, c_6) = 4 \quad d_H(c_5, c_6) = 2$$

Assign the vertex v_1 to code c_1 with degree 5 and vertices v_2, v_3, v_4, v_5 and v_6 corresponding to the code words c_1, c_2, c_3, c_4 and c_5 respectively, each having vertex degree 3.
Since, $d_H(c_3, c_6) = 4 = d_H(c_4, c_6)$, the vertices v_3 and v_4 cannot be adjacent to the vertex v_6. The vertex v_1 is adjacent to all other five vertices. Thus the graph G_2 from C_2 is :

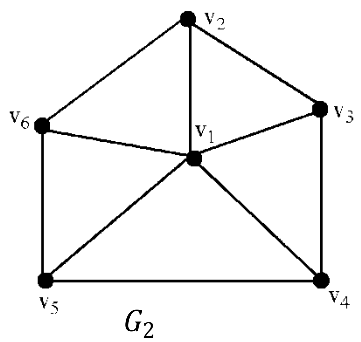
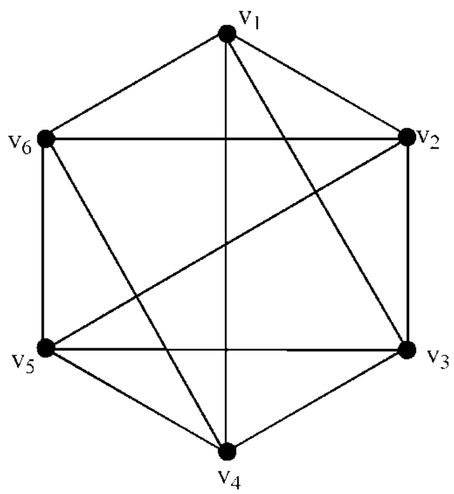


Figure 4

Example 5
Consider the code
$$C_3 = \{111100, 111010, 011110, 011101, 011011, 111001\}.$$

The degree sequence is graphical, the expected graph G_3 is four regular of order 6. Let the code words be called $c_1, c_2, \ldots, c_6$ in the respective order in C_3. If $v_1, v_2, \ldots, v_6$ are the vertices with respect to the code words, from the degree- distance criteria (after finding the distance between every pair of code words and considering their respective Hamming weights) it follows that v_1 not adjacent to v_5 , v_2 not adjacent to v_4 and v_3 not adjacent to v_6. Thus G_3 with respect to C_3 is:



G_3

Figure 5

Example 6

Consider the code $C_4 = \begin{Bmatrix} 010101, 110010, 100110, 111000, 111110, \\ 010110, 110001, 100011, 110100, 110111 \end{Bmatrix}$

$W_{C_4}(x) = 8x^3 + 2x^5$, which implies, if a graph G_4 exists for C_4 of its 10 vertices, two are of degree 5 each and rest are of degree 3. Also,

$d_H(010101,\ 110010) = 4$ $\quad$ $d_H(010101,\ 100110) = 4$

$d_H(010101, 111000\ ) = 4$ $\quad$ $d_H(010101,\ 111110) = 4$

$d_H(010101,\ \ 010110) = 2$ $\quad$ $d_H(010101,\ 110001) = 2$

$d_H(010101, 100011\ ) = 4$ $\quad$ $d_H(010101,\ 110100\ ) = 2$

$d_H(010101, 110111\ ) = 2$ $\quad$ $d_H(110010, 100110) = 2$

$d_H(110010, 111000) = 2$ $\quad$ $d_H(110010, 111110) = 2$

$d_H(110010,\ 010110) = 2$ $\quad$ $d_H(110010,\ 110001) = 2$

$d_H(110010, 100011) = 2$ $\quad$ $d_H(110010, 110100) = 2$

$d_H(110010, 110111) = 2$ $\quad$ $d_H(100110, 111000) = 4$

$d_H(100110, 111110) = 2$ $\quad$ $d_H(100110, 010110) = 2$

$d_H(100110, 110001) = 4$ $\quad$ $d_H(100110, 100011) = 2$

$d_H(100110, 110100) = 2$ $\quad$ $d_H(100110, 110111) = 2$

$d_H(111000, 111110) = 2$ $\quad$ $d_H(111000, 010110) = 4$

$d_H(111000,110001) = 2$ $\quad$ $d_H(111000,100011) = 4$

$d_H(111000,110100) = 2$ $\quad$ $d_H(111000,110111) = 4$

$d_H(111110, 010110) = 2$ $\quad$ $d_H(111110, 110001) = 4$

$d_H(111110,100011\ ) = 4$ $\quad$ $d_H(111110, 110100) = 2$

$d_H(111110, 110111) = 2$ $\quad$ $d_H(010110, 110001) = 4$

$d_H(010110, 100011) = 4$ $\quad$ $d_H(010110, 110100) = 2$

$d_H(010110,110111\ ) = 2$ $\quad$ $d_H(110001,100011\ ) = 2$

$d_H(110001,110100\ ) = 2$ $\quad$ $d_H(110001, 110111) = 2$

$d_H(100011,110100) = 4$ $\quad$ $d_H(100011,110111) = 2$

$d_H(110100, 110111) = 2$

There are only two code words 111110 and 110111 with Hamming weight 5 and their Hamming distance is 2 ⇒they may represent adjacent vertices. The vertices corresponding to the code words with weight 3 are adjacent to the vertices which are assigned code words with weight 5, if and only if, the Hamming distance between those words are 2. Also, code words with weight 3 can be assigned to adjacent vertices of degree 3, only if their corresponding Hamming distance is 2.

Hence if $v_1, v_2, v_3, \ldots, v_{10}$ are the vertices with respect to the code words in the given order, the probabilities of the adjacencies of each vertex are as follows:

$v_1 \ \rightarrow v_6, v_7, v_9, v_{10}.$

$v_2 \rightarrow v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}.$

$v_3 \rightarrow v_2, v_5, v_6, v_8, v_9, v_{10}.$

$v_4 \rightarrow v_2, v_5, v_7, v_9.$

$v_5 \rightarrow v_2, v_3, v_4, v_6, v_9, v_{10}.$

$v_6 \rightarrow v_1, v_2, v_3, v_5, v_9, v_{10}.$

$v_7 \rightarrow v_1, v_2, v_4, v_8, v_9, v_{10}.$

$v_8 \rightarrow v_2, v_3, v_7, v_{10}$.

$v_9 \rightarrow v_1, v_2, v_3, v_4, v_5, v_6, v_9, v_{10}.$

$v_{10} \rightarrow v_1, v_2, v_3, v_5, v_6, v_7, v_8, v_9$.

Of the many possible representations of G_4 from C_4 three are represented as follows.

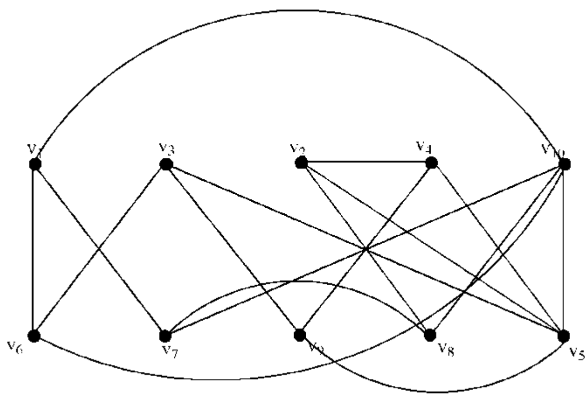


Figure 6

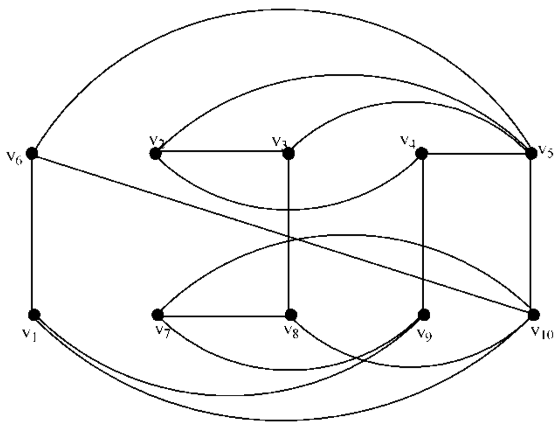


Figure 7

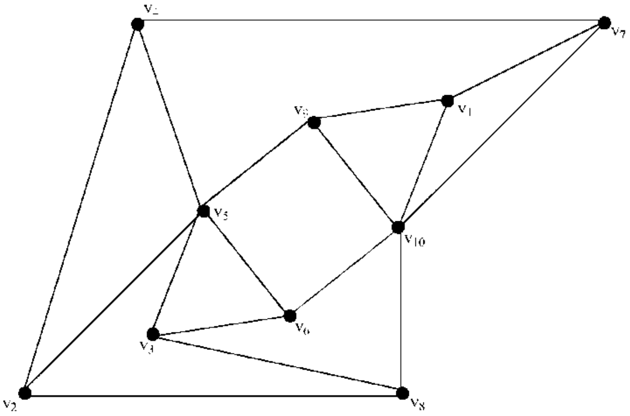


Figure 8

Remark

As the code words of the vertex code of a particular graph may differ, the graphical representation of a given code too will differ. Figures 6 and 7 are isomorphic representations of code C_4, where as figure 8 also represents C_4, and is non isomorphic to Figures 6 and 7.

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