Double series relations involving multivariable Gimel-function

Frédéric Ayant

Teacher in High School , France

ABSTRACT.
Paliwal [5] given four double series relations about the A-function of n variables, see Gautam and Goyal [3,4]. In this paper, we have obtained some doubles series relations concerning the generalized multivariable Gimel-function defined here.

KEYWORDS : Generalized multivariable Gimel-function, multiple integral contours, double series relations.

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1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C},\mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. We define a generalized transcendental function of several complex variables.

$$\gimel(z_1,\cdots,z_r)=\gimel^{m_2,n_2;m_3,n_3;\cdots;m_r,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c} z_1\\ \cdot\\ \cdot\\ \cdot\\ z_r\end{array}\right|$$

$$[(a_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},$$
$$[(b_{2j};\beta^{(1)}_{2j},\beta^{(2)}_{2j};B_{2j})]_{1,m_2},[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{m_2+1,q_{i_2}},[(b_{3j};\beta^{(1)}_{3j},\beta^{(2)}_{3j},\beta^{(3)}_{3j};B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(a_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{m_3+1,q_{i_3}};\cdots;[(b_{rj};\beta^{(1)}_{rj},\cdots,\beta^{(r)}_{rj};B_{rj})_{1,m_r}],$$

$$[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{n_r+1,p_r}]:\ [(c^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]$$
$$[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{m_r+1,q_r}]:[(d^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]$$

$$\left.\begin{array}{l};\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_j)_{n^{(r)}+1,p^{(r)}_i}]\\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_j)_{m^{(r)}+1,q^{(r)}_i}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{m_2}\Gamma^{B_{2j}}(b_{2j}-\sum_{k=1}^{2}\beta^{(k)}_{2j}s_k)\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha^{(k)}_{2j}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha^{(k)}_{2ji_2}s_k)\prod_{j=m_2+1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta^{(k)}_{2ji2}s_k)]}$$

$$\frac{\prod_{j=1}^{m_3}\Gamma^{B_{3j}}(b_{3j}-\sum_{k=1}^{3}\beta^{(k)}_{3j}s_k)\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha^{(k)}_{3j}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha^{(k)}_{3ji_3}s_k)\prod_{j=m_3+1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta^{(k)}_{3ji3}s_k)]}$$

. . . .

$$\frac{\prod_{j=1}^{m_r}\Gamma^{B_{rj}}(b_{rj}-\sum_{k=1}^{r}\beta_{rj}^{(k)}s_k)\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=m_r+1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)})]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $m_2,n_2,\cdots,m_r,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2\leqslant q_{i_2},0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant m_r\leqslant q_{i_r},0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m_k);(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kj}^{(l)},B_{kj}\in\mathbb{R}^+;(j=1,\cdots,m_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m_k+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n_k+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n_k);(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m_k);(k=1,\cdots,r)$.

$a_{kji_k}\in\mathbb{C};(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r)$.

$b_{kji_k}\in\mathbb{C};(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r)$.

$d_{ji^{(k)}}^{(k)}\in\mathbb{C};(i=1,\cdots,R^{(k)});(j=m_k+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{C};(i=1,\cdots,R^{(k)});(j=n_k+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

The contour L_k is in the $s_k(k=1,\cdots,r)$- plane and run from $\sigma-i\infty$ to $\sigma+i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k\right)(j=1,\cdots,n_2),\Gamma^{A_{3j}}\left(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k\right)$

$(j=1,\cdots,n_3),\cdots,\Gamma^{A_{rj}}\left(1-a_{rj}+\sum_{i=1}^{r}\alpha_{rj}^{(i)}\right)(j=1,\cdots,n_r)$, $\Gamma^{C_j^{(k)}}\left(1-c_j^{(k)}+\gamma_j^{(k)}s_k\right)(j=1,\cdots,n^{(k)})(k=1,\cdots,r)$to the right of the contour L_k and the poles of $\Gamma^{B_{2j}}\left(b_{2j}-\sum_{k=1}^{2}\beta_{2j}^{(k)}s_k\right)(j=1,\cdots,m_2),\Gamma^{B_{3j}}\left(b_{3j}-\sum_{k=1}^{3}\beta_{3j}^{(k)}s_k\right)(j=1,\cdots,m_3)$ $,\cdots,\Gamma^{B_{rj}}\left(b_{rj}-\sum_{i=1}^{r}\beta_{rj}^{(i)}\right)(j=1,\cdots,m_r)$, $\Gamma^{D_j^{(k)}}\left(d_j^{(k)}-\delta_j^{(k)}s_k\right)(j=1,\cdots,m^{(k)})(k=1,\cdots,r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)|<\frac{1}{2}A_i^{(k)}\pi$ where

$$A_i^{(k)}=\sum_{j=1}^{m^{(k)}}D_j^{(k)}\delta_j^{(k)}+\sum_{j=1}^{n^{(k)}}C_j^{(k)}\gamma_j^{(k)}-\tau_{i^{(k)}}\left(\sum_{j=m^{(k)}+1}^{q_i^{(k)}}D_{ji^{(k)}}^{(k)}\delta_{ji^{(k)}}^{(k)}+\sum_{j=n^{(k)}+1}^{p_i^{(k)}}C_{ji^{(k)}}^{(k)}\gamma_{ji^{(k)}}^{(k)}\right)+$$

$$\sum_{j=1}^{n_2}A_{2j}\alpha_{2j}^{(k)}+\sum_{j=1}^{m_2}B_{2j}\beta_{2j}^{(k)}-\tau_{i_2}\left(\sum_{j=n_2+1}^{p_{i_2}}A_{2ji_2}\alpha_{2ji_2}^{(k)}+\sum_{j=m_2+1}^{q_{i_2}}B_{2ji_2}\beta_{2ji_2}^{(k)}\right)+\cdots+$$

$$\sum_{j=1}^{n_r}A_{rj}\alpha_{rj}^{(k)}+\sum_{j=1}^{m_r}B_{rj}\beta_{rj}^{(k)}-\tau_{i_r}\left(\sum_{j=n_r+1}^{p_{i_r}}A_{rji_r}\alpha_{rji_r}^{(k)}+\sum_{j=m_r+1}^{q_{i_r}}B_{rji_r}\beta_{rji_r}^{(k)}\right) \quad (1.4)$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,),max(\,|z_1|,\cdots,|z_r|\,)\to 0$

$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,),min(\,|z_1|,\cdots,|z_r|\,)\to\infty$ where $i=1,\cdots,r$:

$\alpha_i=\min_{\substack{1\leqslant j\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)$ and $\beta_i=\max_{\substack{1\leqslant j\leqslant n_i\\1\leqslant j\leqslant n^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}A_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}}+C_j^{(i)}\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)$

Remark 1.
If $m_2=n_2=\cdots=m_{r-1}=n_{r-1}=p_{i_2}=q_{i_2}=\cdots=p_{i_{r-1}}=q_{i_{r-1}}=0$ and $A_{2j}=B_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=B_{rj}=A_{rji_r}=B_{rji_r}=1$, then the generalized multivariable Gimel-function reduces in the generalized multivariable Aleph- function (extension of multivariable Aleph-function defined by Ayant [1]).

Remark 2.
If $m_2=n_2=\cdots=m_r=n_r=p_{i_2}=q_{i_2}=\cdots=p_{i_r}=q_{i_r}=0$ and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2==\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then the generalized multivariable Gimel-function reduces in a generalized multivariable I-function (extension of multivariable I-function defined by Prathima et al. [7]).

Remark 3.
If$A_{2j}=B_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=B_{rj}=A_{rji_r}=B_{rji_r}=1$and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2=\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then the generalized multivariable Gimel-function reduces in generalized of multivariable I-function (extension of multivariable I-function defined by Prasad [6]).

Remark 4.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the generalized multivariable H-function (extension of multivariable H-function defined by Srivastava and Panda [9,10]).

In your investigation, we shall use the following notations.

$\mathbb{A}=[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.5}$$

$$\mathbf{A}=[(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A=[(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,n^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_j^{(r)})_{n^{(r)}+1,p_i^{(r)}}] \tag{1.7}$$

$$\mathbb{B}=[(b_{2j};\beta_{2j}^{(1)},\beta_{2j}^{(2)};B_{2j})]_{1,m_2},[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{m_2+1,q_{i_2}},[(b_{3j};\beta_{3j}^{(1)},\beta_{3j}^{(2)},\beta_{3j}^{(3)};B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{m_3+1,q_{i_3}};\cdots;[(\mathrm{b}_{(r-1)j};\beta_{(r-1)j}^{(1)},\cdots,\beta_{(r-1)j}^{((r-1)j)};B_{(r-1)j})_{1,m_{r-1}}],$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})_{m_{r-1}+1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B}=[(\mathrm{b}_{rj};\beta_{rj}^{(1)},\cdots,\beta_{rj}^{(r)};B_{rj})_{1,m_r}],[\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{m_r+1,q_{i_r}}] \tag{1.9}$$

$$\mathrm{B}=[(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}];\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_j^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \tag{1.10}$$

$$U=m_2,n_2;m_3,n_3;\cdots;m_{r-1},n_{r-1};V=m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X=p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

2. Double series relations.

In this section, we establish four double series relations about the generalized multivariable Gimel-function.

Theorem 1.

$$\sum_{u,v=0}^{\infty}\frac{(b)_{u+v}(c)_{u+v}}{(1+a-d-e)_{u+v}}\gimel_{X;p_{i_r}+4,q_{i_r}+3,\tau_{i_r}:R_r:Y}^{U;m_r+4,n_r:V}$$

$$\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};\ (\mathrm{a}+\mathrm{u}+\mathrm{v};\mathrm{a}_1,\cdots,a_r;1),(1+\frac{a}{2}+u+v;\frac{a_1}{2},\cdots,\frac{a_r}{2};1),(d+u;\frac{a_1}{2},\cdots,\frac{a_r}{2};1),(e+v;\frac{a_1}{2},\cdots,\frac{a_r}{2};1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\ (\frac{a}{2}+\mathrm{u}+\mathrm{v};\frac{a_1}{2},\cdots,\frac{a_r}{2};1),(1+a-b+u+v;a_1,\cdots,a_r;1),(1+a-c+u+v;a_1,\cdots,a_r;1),\mathbf{B}:B \end{array}\right)$$

$$=\frac{\Gamma(1+a-d-e)\Gamma(1+a-b-c-d-e)}{2\Gamma(1+a-b-d-e)\Gamma(1+a-c-d-e)}$$

$$\gimel^{U;m_r+2,n_r:V}_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1 & \mathbb{A};\ (d;\frac{a_1}{2},\cdots,\frac{a_r}{2};1),(e;\frac{a_1}{2},\cdots,\frac{a_r}{2};1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\ (1+a-b-c;a_1,\cdots,a_r;1),\mathbf{B}:B \end{array}\right) \tag{2.1}$$

provided that

$a_i > 0 (i = 1,\cdots,r), Re(b+c+d+e-a) < 1$ and $|arg(z_k)| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4) and the double series involved on the left-hand side of (2.1) is absolutely convergent.

Proof
To establish (2.1), expressing the generalized multivariable Gimel-function on the left-hand side of (2.1) in terms of Mellin-Barnes multiple integrals contour with the help of (1.1), changing the order of integrations and summations (as the double series involved is absolutely convergent) and then evaluating the inner series with the help of the following result of Sharma ([8], p. 187)

$$\sum_{u,v=0}^{\infty}\frac{(a)_{u+v}\left(\frac{a}{2}+1\right)_{u+v}(b)_{u+v}(c)_{u+v}(d)_u(e)_v}{\left(\frac{a}{2}\right)_{u+v}(1+a-b)_{u+v}(1+a-c)_{u+v}(1+a-d-e)_{u+v}u!v!}=$$

$$\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d-e)\Gamma(1+a-b-c-d-e)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d-e)\Gamma(1+a-c-d-e)} \tag{2.2}$$

where $Re(b+c+d+e-a) < 1$, and interpreting the Mellin-Barnes multiple integrals contour in terms of the generalized multivariable Gimel-function with the help of (1.1) , we get the desired result (2.1).

Theorem 2.

$$\sum_{u,v=0}^{\infty}\frac{(d)_u(e)_v}{u!v!}\gimel^{U;m_r+4,n_r:V}_{X;p_{i_r}+4,q_{i_r}+4,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} z_1 & \mathbb{A};\mathbf{A},\ (a+u+v;a_1,\cdots,a_r;1),(1+\frac{a}{2}+u+v;\frac{a_1}{2},\cdots,\frac{a_r}{2};1),(b+u+v;\frac{a_1}{2},\cdots,\frac{a_r}{2};1),(c+u+v;\frac{a_1}{2},\cdots,\frac{a_r}{2};1):A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\ (\frac{a}{2}+u+v;\frac{a_1}{2},\cdots,\frac{a_r}{2};1),(1+a-b+u+v;\frac{a_1}{2},\cdots,\frac{a_r}{2};1),(1+a-c+u+v;\frac{a_1}{2},\cdots,\frac{a_r}{2};1),(1+a-d+u+v;\frac{a_1}{2},\cdots,\frac{a_r}{2};1),\mathbf{B}:B \end{array}\right)$$

$$=\frac{\Gamma(1+a-b-d-e)}{2\Gamma(1+a-b-c)}\gimel^{U;m_r+2,n_r:V}_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} z_1 & \mathbb{A};\ \mathbf{A},(b;\frac{a_1}{2},\cdots,\frac{a_r}{2};1),(c;\frac{a_1}{2},\cdots,\frac{a_r}{2};1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\ (1+a-b-d-e;a_1,\cdots,a_r;1),(1+a-c-d-e;a_1,\cdots,a_r;1),\mathbf{B}:B \end{array}\right) \tag{2.3}$$

provided that

$a_i > 0 (i = 1,\cdots,r), Re(1+a-b-c-d-e) > 0$ and $|arg(z_k)| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4) and the double series involved on the left-hand side of (2.3) is absolutely convergent.

Theorem 3.

$$\sum_{u,v=0}^{\infty}\frac{(-)^{u+v}(d)_u(e)_v}{(1+a-b)_{u+v}u!v!}\gimel^{U;m_r+3,n_r:V}_{X;p_{i_r}+3,q_{i_r}+2,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} z_1 & \mathbb{A};\ (\mathrm{d{+}u{+}v};a_1,\cdots,a_r;1),(1+\frac{a}{2}+u+v;\frac{a_1}{2},\cdots,\frac{a_r}{2};1),(b+u+v;a_1,\cdots,a_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\ (\frac{a}{2}{+}\mathrm{u{+}v};\frac{a_1}{2},\cdots,\frac{a_r}{2};1),(1+a-d-e+u+v;a_1,\cdots,a_r;1),\mathbf{B}:B \end{array}\right)$$

$$= \frac{\Gamma(1+a-b)}{\Gamma(1+a-b-d-e)} \gimel^{U;m_r+1,n_r:V}_{X;p_{i_r}+1,q_{i_r},\tau_{i_r}:R_r:Y} \left(\begin{array}{c|c} z_1 & \mathbb{A};\ \mathbf{A},(\mathrm{b};a_1,\cdots,a_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\ \mathbf{B}:\ \mathrm{B} \end{array}\right) \tag{2.4}$$

provided that

$a_i > 0 (i = 1, \cdots, r)$, and $|arg(z_k)| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4) and the double series involved on the left-hand side of (2.4) is absolutely convergent.

Theorem 4.

$$\sum_{u,v=0}^{\infty} \frac{(-)^{u+v}(b)_{u+v}}{(1+a-d-e)_{u+v}u!v!} \gimel^{U;m_r+4,n_r:V}_{X;p_{i_r}+4,q_{i_r}+2,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} z_1 & \mathbb{A};\ (\mathrm{a{+}u{+}v};a_1,\cdots,a_r;1),(1+\frac{a}{2}+u+v;\frac{a_1}{2},\cdots,\frac{a_r}{2};1),(d+u;\frac{a_1}{2},\cdots,\frac{a_r}{2};1),(e+v;\frac{a_1}{2},\cdots,\frac{a_r}{2};1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\ (\frac{a}{2}{+}\mathrm{u{+}v};\frac{a_1}{2},\cdots,\frac{a_r}{2};1),(1+a-b+u+v;a_1,\cdots,a_r;1),\mathbf{B}:B \end{array}\right)$$

$$= \frac{\Gamma(1+a-d-e)}{2\Gamma(1+a-b-d-e)} \gimel^{U;m_r+2,n_r:V}_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y} \left(\begin{array}{c|c} z_1 & \mathbb{A};\ (\mathrm{d};\frac{a_1}{2},\cdots,\frac{a_r}{2};1),(e;\frac{a_1}{2},\cdots,\frac{a_r}{2};1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\ \mathbf{B}:\ \mathrm{B} \end{array}\right) \tag{2.5}$$

provided that

$a_i > 0 (i = 1, \cdots, r)$, and $|arg(z_k)| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4) and the double series involved on the left-hand side of (2.5) is absolutely convergent.

The proofs of theorems 2, 3 and 4 can be developped on similar lines to those given using (2.2) and the following formula due to Sharma ([8], p. 185).

$$\sum_{u,v=0}^{\infty} \frac{(-)^{u+v}(a)_{u+v}\left(\frac{a}{2}+1\right)_{u+v}(b)_{u+v}(d)_u(e)_v}{\left(\frac{a}{2}\right)_{u+v}(1+a-b)_{u+v}(1+a-d-e)_{u+v}u!v!} = \frac{\Gamma(1+a-b)\Gamma(1+a-d-e)}{\Gamma(1+a)\Gamma(1+a-b-d-e)} \tag{2.6}$$

provided $Re\left(\frac{a}{2}-b-d-e\right) > 1.$

5. Conclusion.

The double series relations established here are unified and act as key formulae. Thus the generalized multivariable Gimel-function occurring in these formulae can be suitably specialized to a remarkably wide variety of special functions (or product of several such special functions) which are expressible in terms of E, F, G, H, I, Aleph-functions of one or more variables.

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