# On Integration with respect to their parameters 

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## ABSTRACT

In the present paper some interesting results have been obtained on integrating the multivariable Gimel-function defined in this document with respect to its parameters. Results obtained are important in connection with the study of the problems of applied Mathematics and certain boundary value problems.

KEYWORDS : Multivariable Gimel-function, multiple integral contours, integration with respect to a parameter.
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## 1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C}, \mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_{0}=\mathbb{N} \cup\{0\}$. We define a generalized transcendental function of several complex variables.

$$
\begin{gathered}
{\left[\left(a_{2 j} ; \alpha_{2 j}^{(1)}, \alpha_{2 j}^{(2)} ; A_{2 j}\right)\right]_{1, n_{2}},\left[\tau_{i_{2}}\left(a_{2 j i_{2}} ; \alpha_{2 j i_{2}}^{(1)}, \alpha_{2 j i_{2}}^{(2)} ; A_{2 j i_{2}}\right)\right]_{n_{2}+1, p_{i_{2}}} ;\left[\left(a_{3 j} ; \alpha_{3 j}^{(1)}, \alpha_{3 j}^{(2)}, \alpha_{3 j}^{(3)} ; A_{3 j}\right)\right]_{1, n_{3}},} \\
{\left[\tau_{i_{2}}\left(b_{2 j i_{2}} ; \beta_{2 j i_{2}}, \beta_{2 j i_{2}}^{(2)}, B_{2 j i_{2} 2}\right)\right]_{1, q_{2} i_{2}} ;}
\end{gathered}
$$

$$
\left[\tau_{i_{r}}\left(a_{r j j_{r}} ; \alpha_{r j i_{r},}^{(1)} \cdots, \alpha_{r j i_{r}}^{(r)} ; A_{r j i_{r}}\right)_{\left.n_{r}+1, p_{r}\right]}\right]:\left[\left(c_{j}^{(1)}, \gamma_{j}^{(1)} ; C_{j}^{(1)}\right)_{1, n} n^{(1)}\right],\left[\tau_{i^{(1)}}\left(c_{j i^{(1)}}^{(1)}, \gamma_{j i^{(1)}}^{(1)} ; C_{j i^{(1)}}^{(1)}\right)_{n^{(1)}+1, p_{i}^{(1)}}\right]
$$

$$
\left[\tau_{i_{r}}\left(b_{r j i_{r}} ; \beta_{r j i_{r}}^{(1)}, \cdots, \beta_{r j i_{r}}^{(r)} ; B_{\left.r j i_{r}\right)}^{\left.r_{1}, q_{r}\right]}\right]: \quad\left[\left(\mathrm{d}_{j}^{(1)}\right), \delta_{j}^{(1)} ; D_{j}^{(1)}\right)_{1, m(1)}^{(1)}\right],\left[\tau_{i(1)}^{(1)}\left(d_{j i^{(1)}}^{(1)}, \delta_{j i}^{(1)} ; D_{j i^{(1)}}^{(1)}\right)_{m(1)+1, q_{i}^{(1)}}^{(1)}\right]
$$

$$
\begin{equation*}
=\frac{1}{(2 \pi \omega)^{r}} \int_{L_{1}} \cdots \int_{L_{r}} \psi\left(s_{1}, \cdots, s_{r}\right) \prod_{k=1}^{r} \theta_{k}\left(s_{k}\right) z_{k}^{s_{k}} \mathrm{~d} s_{1} \cdots \mathrm{~d} s_{r} \tag{1.1}
\end{equation*}
$$

with $\omega=\sqrt{-1}$

$$
\begin{aligned}
\psi\left(s_{1}, \cdots, s_{r}\right)= & \frac{\prod_{j=1}^{n_{2}} \Gamma^{A_{2 j}}\left(1-a_{2 j}+\sum_{k=1}^{2} \alpha_{2 j}^{(k)} s_{k}\right)}{\sum_{i_{2}=1}^{R_{2}}\left[\tau_{i_{2}} \prod_{j=n_{2}+1}^{p_{i}} \Gamma^{A_{2 j i_{2}}}\left(a_{2 j i_{2}}-\sum_{k=1}^{2} \alpha_{2 j i_{2}}^{(k)} s_{k}\right) \prod_{j=1}^{q_{i}} \Gamma^{B_{2 j i_{2}}}\left(1-b_{2 j i 2}+\sum_{k=1}^{2} \beta_{2 j i 2}^{(k)} s_{k}\right)\right]} \\
& \frac{\prod_{j=1}^{n_{3}} \Gamma^{A_{3 j}}\left(1-a_{3 j}+\sum_{k=1}^{3} \alpha_{3 j}^{(k)} s_{k}\right)}{\sum_{i_{3}=1}^{R_{3}}\left[\tau_{i_{3}} \prod_{j=n_{3}+1}^{p_{i_{3}}} \Gamma^{A_{3 j i_{3}}}\left(a_{3 j i_{3}}-\sum_{k=1}^{3} \alpha_{3 j i_{3}}^{k} s_{k}\right) \prod_{j=1}^{q_{i 3}} \Gamma^{B_{3 j i_{3}}}\left(1-b_{3 j i 3}+\sum_{k=1}^{3} \beta_{3 j i_{3}}^{(k)} s_{k}\right)\right]}
\end{aligned}
$$

$$
\begin{aligned}
& {\left[\tau_{i_{3}}\left(a_{3 j i_{3}} ; \alpha_{\left.3 i_{3}\right)}^{(1)}, \alpha_{3 j j_{3}}^{(2)}, \alpha_{3 j i_{3}}^{(3)} ; A_{3 j i_{3}}\right)\right]_{n_{3}+1, p_{i_{3}}} ; \cdots ;\left[\left(a_{r j} ; \alpha_{r j}^{(1)}, \cdots, \alpha_{r j}^{(r)} ; A_{r j}\right)_{1, n_{r}}\right],} \\
& {\left[\tau_{i_{3}}\left(b_{3 j i_{3}} ; \beta_{3 i_{3}}^{(1)}, \beta_{3 j i_{3}}^{(2)}, \beta_{3 j i_{3}}^{(3)} ; B_{3 j i_{3}}\right)\right]_{1, q_{i_{3}}} ; \cdots ;}
\end{aligned}
$$

$$
\begin{equation*}
\frac{\prod_{j=1}^{n_{r}} \Gamma^{A_{r j}}\left(1-a_{r j}+\sum_{k=1}^{r} \alpha_{r j}^{(k)} s_{k}\right)}{\sum_{i_{r}=1}^{R_{r}}\left[\tau_{i_{r}} \prod_{j=n_{r}+1}^{p_{i_{r}}} \Gamma^{A_{r j j_{r} r}}\left(a_{r j i_{i}}-\sum_{k=1}^{r} \alpha_{r j i_{r}}^{(k)} s_{k}\right) \prod_{j=1}^{q_{i r}} \Gamma^{B_{r j i_{r} r}}\left(1-b_{r j i r}+\sum_{k=1}^{r} \beta_{r j i r}^{(k)} s_{k}\right)\right]} \tag{1.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\theta_{k}\left(s_{k}\right)=\frac{\prod_{j=1}^{m^{(k)}} \Gamma^{D_{j}^{(k)}}\left(d_{j}^{(k)}-\delta_{j}^{(k)} s_{k}\right) \prod_{j=1}^{n^{(k)}} \Gamma^{C_{j}^{(k)}}\left(1-c_{j}^{(k)}+\gamma_{j}^{(k)} s_{k}\right)}{\sum_{i^{(k)}=1}^{R^{(k)}}\left[\tau_{i^{(k)}} \prod_{j=m^{(k)}+1}^{q_{i}(k)} \Gamma^{D_{j i}^{(k)}}\left(1-d_{j i(k)}^{(k)}+\delta_{j i(k)}^{(k)} s_{k}\right) \prod_{j=n^{(k)}+1}^{p_{i(k)}} \Gamma^{C_{j i}^{(k)}}\left(c_{j i(k)}^{(k)}-\gamma_{j i(k)}^{(k)} s_{k}\right)\right]} \tag{1.3}
\end{equation*}
$$

1) $\left[\left(c_{j}^{(1)} ; \gamma_{j}^{(1)}\right]_{1, n_{1}}\right.$ stands for $\left(c_{1}^{(1)} ; \gamma_{1}^{(1)}\right), \cdots,\left(c_{n_{1}}^{(1)} ; \gamma_{n_{1}}^{(1)}\right)$.
2) $n_{2}, \cdots, n_{r}, m^{(1)}, n^{(1)}, \cdots, m^{(r)}, n^{(r)}, p_{i_{2}}, q_{i_{2}}, R_{2}, \tau_{i_{2}}, \cdots, p_{i_{r}}, q_{i_{r}}, R_{r}, \tau_{i_{r}}, p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i(r)}, R^{(r)} \in \mathbb{N}$ and verify : $0 \leqslant m_{2}, 0 \leqslant n_{2} \leqslant p_{i_{2}}, \cdots, 0 \leqslant m_{r}, 0 \leqslant n_{r} \leqslant p_{i_{r}}, 0 \leqslant m^{(1)} \leqslant q_{i^{(1)}}, \cdots, 0 \leqslant m^{(r)} \leqslant q_{i^{(r)}}$.
3) $\tau_{i_{2}}\left(i_{2}=1, \cdots, R_{2}\right) \in \mathbb{R}^{+} ; \tau_{i_{r}} \in \mathbb{R}^{+}\left(i_{r}=1, \cdots, R_{r}\right) ; \tau_{i^{(k)}} \in \mathbb{R}^{+}\left(i=1, \cdots, R^{(k)}\right),(k=1, \cdots, r)$.
4) $\gamma_{j}^{(k)}, C_{j}^{(k)} \in \mathbb{R}^{+} ;\left(j=1, \cdots, n_{k}\right) ;(k=1, \cdots, r) ; \delta_{j}^{(k)}, D_{j}^{(k)} \in \mathbb{R}^{+} ;\left(j=1, \cdots, m_{k}\right) ;(k=1, \cdots, r)$.
$\alpha_{k j}^{(l)}, A_{k j} \in \mathbb{R}^{+} ;\left(j=1, \cdots, n_{k}\right) ;(k=2, \cdots, r) ;(l=1, \cdots, k)$.
$\alpha_{k j i_{k}}^{(l)}, A_{k j i_{k}} \in \mathbb{R}^{+} ;\left(j=n_{k}+1, \cdots, p_{i_{k}}\right) ;(k=2, \cdots, r) ;(l=1, \cdots, k)$.
$\beta_{k j i_{k}}^{(l)}, B_{k j i_{k}} \in \mathbb{R}^{+} ;\left(j=m_{k}+1, \cdots, q_{i_{k}}\right) ;(k=2, \cdots, r) ;(l=1, \cdots, k)$.
$\delta_{j i(k)}^{(k)} \in \mathbb{R}^{+} ;\left(i=1, \cdots, R^{(k)}\right) ;\left(j=m^{(k)}+1, \cdots, q_{i(k)}\right) ;(k=1, \cdots, r)$.
$\gamma_{j i(k)}^{(k)} \in \mathbb{R}^{+} ;\left(i=1, \cdots, R^{(k)}\right) ;\left(j=n^{(k)}+1, \cdots, p_{i^{(k)}}\right) ;(k=1, \cdots, r)$.
5) $c_{j}^{(k)} \in \mathbb{C} ;\left(j=1, \cdots, n_{k}\right) ;(k=1, \cdots, r) ; d_{j}^{(k)} \in \mathbb{C} ;\left(j=1, \cdots, m_{k}\right) ;(k=1, \cdots, r)$.
$a_{k j i_{k}} \in \mathbb{C} ;\left(j=n_{k}+1, \cdots, p_{i_{k}}\right) ;(k=2, \cdots, r)$.
$b_{k j i_{k}} \in \mathbb{C} ;\left(j=m_{k}+1, \cdots, q_{i_{k}}\right) ;(k=2, \cdots, r)$.
$d_{j i^{(k)}}^{(k)} \in \mathbb{C} ;\left(i=1, \cdots, R^{(k)}\right) ;\left(j=m^{(k)}+1, \cdots, q_{i^{(k)}}\right) ;(k=1, \cdots, r)$.
$\gamma_{j i(k)}^{(k)} \in \mathbb{C} ;\left(i=1, \cdots, R^{(k)}\right) ;\left(j=n^{(k)}+1, \cdots, p_{i(k)}\right) ;(k=1, \cdots, r)$.
The contour $L_{k}$ is in the $s_{k}(k=1, \cdots, r)$ - plane and run from $\sigma-i \infty$ to $\sigma+i \infty$ where $\sigma$ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2} j}\left(1-a_{2 j}+\sum_{k=1}^{2} \alpha_{2 j}^{(k)} s_{k}\right)\left(j=1, \cdots, n_{2}\right), \Gamma^{A_{3} j}\left(1-a_{3 j}+\sum_{k=1}^{3} \alpha_{3 j}^{(k)} s_{k}\right)$
$\left(j=1, \cdots, n_{3}\right), \cdots, \Gamma^{A_{r j}}\left(1-a_{r j}+\sum_{i=1}^{r} \alpha_{r j}^{(i)}\right)\left(j=1, \cdots, n_{r}\right), \Gamma_{j}^{C_{j}^{(k)}}\left(1-c_{j}^{(k)}+\gamma_{j}^{(k)} s_{k}\right)\left(j=1, \cdots, n^{(k)}\right)(k=1, \cdots, r)$ to the right of the contour $L_{k}$ and the poles of $\Gamma^{D_{j}^{(k)}}\left(d_{j}^{(k)}-\delta_{j}^{(k)} s_{k}\right)\left(j=1, \cdots, m^{(k)}\right)(k=1, \cdots, r)$ lie to the left of the contour $L_{k}$. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H -function given by as :
$\left|\arg \left(z_{k}\right)\right|<\frac{1}{2} A_{i}^{(k)} \pi$ where
$A_{i}^{(k)}=\sum_{j=1}^{m^{(k)}} D_{j}^{(k)} \delta_{j}^{(k)}+\sum_{j=1}^{n^{(k)}} C_{j}^{(k)} \gamma_{j}^{(k)}-\tau_{i^{(k)}}\left(\sum_{j=m^{(k)}+1}^{q_{i}^{(k)}} D_{j i^{(k)}}^{(k)} \delta_{j i^{(k)}}^{(k)}+\sum_{j=n^{(k)}+1}^{p_{i}^{(k)}} C_{j i}^{(k)} \gamma_{j i}^{(k)}\right)$
$-\tau_{i_{2}}\left(\sum_{j=n_{2}+1}^{p_{i_{2}}} A_{2 j i_{2}} \alpha_{2 j i_{2}}^{(k)}+\sum_{j=1}^{q_{i_{2}}} B_{2 j i_{2}} \beta_{2 j i_{2}}^{(k)}\right)-\cdots-\tau_{i_{r}}\left(\sum_{j=n_{r}+1}^{p_{i_{r}}} A_{r j i_{r}} \alpha_{r j i_{r}}^{(k)}+\sum_{j=1}^{q_{i_{r}}} B_{r j i_{r}} \beta_{r j i_{r}}^{(k)}\right)$
Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :
$\aleph\left(z_{1}, \cdots, z_{r}\right)=0\left(\left|z_{1}\right|^{\alpha_{1}}, \cdots,\left|z_{r}\right|^{\alpha_{r}}\right), \max \left(\left|z_{1}\right|, \cdots,\left|z_{r}\right|\right) \rightarrow 0$
$\aleph\left(z_{1}, \cdots, z_{r}\right)=0\left(\left|z_{1}\right|^{\beta_{1}}, \cdots,\left|z_{r}\right|^{\beta_{r}}\right), \min \left(\left|z_{1}\right|, \cdots,\left|z_{r}\right|\right) \rightarrow \infty$ where $i=1, \cdots, r:$
$\alpha_{i}=\min _{1 \leqslant j \leqslant m^{(i)}} R e\left[D_{j}^{(i)}\left(\frac{d_{j}^{(i)}}{\delta_{j}^{(i)}}\right)\right]$ and $\beta_{i}=\max _{1 \leqslant j \leqslant n^{(i)}} R e\left[C_{j}^{(i)}\left(\frac{c_{j}^{(i)}-1}{\gamma_{j}^{(i)}}\right)\right]$

## Remark 1.

If $n_{2}=\cdots=n_{r-1}=p_{i_{2}}=q_{i_{2}}=\cdots=p_{i_{r-1}}=q_{i_{r-1}}=0$ and $A_{2 j}=A_{2 j i_{2}}=B_{2 j i_{2}}=\cdots=A_{r j}=A_{r j i_{r}}=B_{r j i_{r}}=1$ $A_{r j}=A_{r j i_{r}}=B_{r j i_{r}}=1$, then the multivariable Gimel-function reduces in the multivariable Aleph- function defined by Ayant [1].

## Remark 2.

If $n_{2}=\cdots=n_{r}=p_{i_{2}}=q_{i_{2}}=\cdots=p_{i_{r}}=q_{i_{r}}=0$ and $\tau_{i_{2}}=\cdots=\tau_{i_{r}}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_{2}=\cdots=R_{r}=R^{(1)}=$ $\cdots=R^{(r)}=1$, then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [7].

## Remark 3.

If $A_{2 j}=A_{2 j i_{2}}=B_{2 j i_{2}}=\cdots=A_{r j}=A_{r j i_{r}}=B_{r j i_{r}}=\operatorname{1and} \tau_{i_{2}}=\cdots=\tau_{i_{r}}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_{2}=\cdots=R_{r}=R^{(1)}$ $=\cdots=R^{(r)}=1$, then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [6].

## Remark 4.

If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H -function defined by Srivastava and Panda [8,9].

In your investigation, we shall use the following notations.
$\mathbb{A}=\left[\left(\mathrm{a}_{2 j} ; \alpha_{2 j}^{(1)}, \alpha_{2 j}^{(2)} ; A_{2 j}\right)\right]_{1, n_{2}},\left[\tau_{i_{2}}\left(a_{2 j i_{2}} ; \alpha_{2 j i_{2}}^{(1)}, \alpha_{2 j i_{2}}^{(2)} ; A_{2 j i_{2}}\right)\right]_{n_{2}+1, p_{i_{2}}},\left[\left(a_{3 j} ; \alpha_{3 j}^{(1)}, \alpha_{3 j}^{(2)}, \alpha_{3 j}^{(3)} ; A_{3 j}\right)\right]_{1, n_{3}}$,
$\left[\tau_{i_{3}}\left(a_{3 j i_{3}} ; \alpha_{3 j i_{3}}^{(1)}, \alpha_{3 j i_{3}}^{(2)}, \alpha_{3 j i_{3}}^{(3)} ; A_{3 j i_{3}}\right)\right]_{n_{3}+1, p_{i_{3}}} ; \cdots ;\left[\left(\mathrm{a}_{(r-1) j} ; \alpha_{(r-1) j}^{(1)}, \cdots, \alpha_{(r-1) j}^{(r-1)} ; A_{(r-1) j}\right)_{1, n_{r-1}}\right]$,
$\left[\tau_{i_{r-1}}\left(a_{(r-1) j i_{r-1}} ; \alpha_{(r-1) j i_{r-1}}^{(1)}, \cdots, \alpha_{(r-1) j i_{r-1}}^{(r-1)} ; A_{(r-1) j i_{r-1}}\right)_{n_{r-1}+1, p_{i_{r-1}}}\right]$
$\mathbf{A}=\left[\left(\mathrm{a}_{r j} ; \alpha_{r j}^{(1)}, \cdots, \alpha_{r j}^{(r)} ; A_{r j}\right)_{1, n_{r}}\right],\left[\tau_{i_{r}}\left(a_{r j i_{r}} ; \alpha_{r j i_{r}}^{(1)}, \cdots, \alpha_{r j i_{r}}^{(r)} ; A_{r j i_{r}}\right)_{\mathfrak{n}+1, p_{i_{r}}}\right]$

$$
\begin{align*}
& A=\left[\left(c_{j}^{(1)}, \gamma_{j}^{(1)} ; C_{j}^{(1)}\right)_{1, n^{(1)}}\right],\left[\tau_{i^{(1)}}\left(c_{j i^{(1)}}^{(1)}, \gamma_{j i^{(1)}}^{(1)} ; C_{j i^{(1)}}^{(1)}\right)_{\left.n^{(1)}+1, p_{i}^{(1)}\right]}\right] ; \cdots ; \\
& {\left[\left(c_{j}^{(r)}, \gamma_{j}^{(r)} ; C_{j}^{(r)}\right)_{1, n(r)}\right],\left[\tau_{i(r)}\left(c_{j i(r)}^{(r)}, \gamma_{j i}^{(r)} ; C_{j}^{(r)}\right)_{n^{(r)}+1, p_{i}^{(r)}}\right]}  \tag{1.7}\\
& \mathbb{B}=\left[\tau_{i_{2}}\left(b_{2 j i_{2}} ; \beta_{2 j i_{2}}^{(1)}, \beta_{2 j i_{2}}^{(2)} ; B_{2 j i_{2}}\right)\right]_{1, q_{i_{2}}},\left[\tau_{i_{3}}\left(b_{3 j i_{3}} ; \beta_{3 j i_{3}}^{(1)}, \beta_{3 j i_{3}}^{(2)}, \beta_{3 j i_{3}}^{(3)} ; B_{3 j i_{3}}\right)\right]_{1, q_{i 3}} ; \cdots ; \\
& {\left[\tau_{i_{r-1}}\left(b_{(r-1) j i_{r-1}} ; \beta_{(r-1) j i_{r-1}}^{(1)}, \cdots, \beta_{(r-1) j i_{r-1}}^{(r-1)} ; B_{(r-1) j i_{r-1}}\right)_{1, q_{i_{r-1}}}\right]} \\
& \mathbf{B}=\left[\tau_{i_{r}}\left(b_{r j i_{r}} ; \beta_{r j i_{r}}^{(1)}, \cdots, \beta_{r j i_{r}}^{(r)} ; B_{r j i_{r}}\right)_{1, q_{i}}\right]  \tag{1.9}\\
& \mathrm{B}=\left[\left(\mathrm{d}_{j}^{(1)}, \delta_{j}^{(1)} ; D_{j}^{(1)}\right)_{1, m^{(1)}}\right],\left[\tau_{i^{(1)}}\left(d_{j i^{(1)}}^{(1)}, \delta_{\left.j i^{1}\right)}^{(1)} ; D_{j i^{(1)}}^{(1)}\right)_{m^{(1)}+1, q_{i}^{(1)}}\right] ; \cdots ; \\
& {\left[\left(\mathrm{d}_{j}^{(r)}, \delta_{j}^{(r)} ; D_{j}^{(r)}\right)_{\left.1, m^{(r)}\right]}\right],\left[\tau_{i}^{(r)}\left(d_{j i}^{(r)}, \delta_{j i^{(r)}}^{(r)} ; D_{j}^{(r)}\right)_{m}^{(r)+1, q_{i}^{(r)}}{ }\right]}  \tag{1.10}\\
& U=0, n_{2} ; 0, n_{3} ; \cdots ; 0, n_{r-1} ; V=m^{(1)}, n^{(1)} ; m^{(2)}, n^{(2)} ; \cdots ; m^{(r)}, n^{(r)}  \tag{1.11}\\
& X=p_{i_{2}}, q_{i_{2}}, \tau_{i_{2}} ; R_{2} ; \cdots ; p_{i_{r-1}}, q_{i_{r-1}}, \tau_{i_{r-1}}: R_{r-1} ; Y=p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}} ; R^{(1)} ; \cdots ; p_{i^{(r)}}, q_{i^{(r)}} ; \tau_{i^{(r)}} ; R^{(r)} \tag{1.12}
\end{align*}
$$

## 2. Required results.

The following known results ([3], p. 300, Eq.(21) and Eq.(22)) will be required

## Lemma 1.

$\int_{-\infty}^{\infty} \frac{\sin \pi x}{\Gamma(a+x) \Gamma(b-x) \Gamma(c+x) \Gamma(d-x)} \mathrm{d} x=\frac{\sin \frac{\pi(b-a)}{2}}{2 \Gamma\left(\frac{a+b}{2}\right) \Gamma\left(\frac{c+d}{2}\right) \Gamma(a+d-1)}$
where $a+d=b+c ; \operatorname{Re}(a+b+c+d)>2$.

## Lemma 2.

$\int_{-\infty}^{\infty} \frac{\cos \pi x}{\Gamma(a+x) \Gamma(b-x) \Gamma(c+x) \Gamma(d-x)} \mathrm{d} x=\frac{\cos \frac{\pi(b-a)}{2}}{2 \Gamma\left(\frac{a+b}{2}\right) \Gamma\left(\frac{c+d}{2}\right) \Gamma(a+d-1)}$
where $a+d=b+c ; \operatorname{Re}(a+b+c+d)>2$.

## 3. Main integrals.

In this section, we evaluate five integrals with respect to their parameters involving the generalized multivariable Gimel-function.

## Theorem 1.

$\int_{-\infty}^{\infty} \sin \pi x \beth_{X ; p_{i_{r}}, q_{i_{r}}+4, \tau_{i_{r}}: R_{r}: Y}^{U ; 0, n_{r}: V}\left(\begin{array}{c|c}\mathrm{z}_{1} & \mathbb{A} ; \mathbf{A}: \mathrm{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_{r} & \mathbb{B} ; \mathbf{B}, \mathrm{B}_{1}: B\end{array}\right) \mathrm{d} x=$
$\frac{\sin \frac{\pi(b-a)}{2}}{2} \mathcal{I}_{X ; p_{i_{r}}, q_{i_{r}}+3, \tau_{i_{r}}: R_{r}: Y}^{U ; 0, n_{r}: V}\left(\begin{array}{c|c}\mathrm{z}_{1} & \mathbb{A} ; \mathbf{A}: \mathrm{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_{r} & \mathbb{B} ; \mathbf{B}, \mathrm{B}_{1}^{\prime}: B\end{array}\right)$
where
$B_{1}=\left(1-a-x ; h_{1}, \cdots, h_{r} ; 1\right),\left(1-b+x ; h_{1}, \cdots, h_{r} ; 1\right),\left(1-c-x ; k_{1}, \cdots, k_{r} ; 1\right),\left(1-d+x ; k_{1}, \cdots, k_{r} ; 1\right)(3$
and
$B_{1}^{\prime}=\left(1-\frac{a+b}{2} ; h_{1}, \cdots, h_{r} ; 1\right),\left(1-\frac{c+d}{2} ; k_{1}, \cdots, k_{r} ; 1\right),\left(2-a-d ; h_{1}+k_{1}, \cdots, h_{r}+k_{r} ; 1\right)$
provided
$h_{i}, k_{i}>0(i=1, \cdots, r), \operatorname{Re}(a+b+c+d)+2 \sum_{i=1}^{r}\left(h_{i}+k_{i}\right) \min _{1 \leqslant j \leqslant m^{(i)}} \operatorname{Re}\left[D_{j}^{(i)}\left(\frac{d_{j}^{(i)}}{\delta_{j}^{(i)}}\right)\right]>2$.
$\left|\arg \left(z_{k}\right)\right|<\frac{1}{2} A_{i}^{(k)} \pi$ where $A_{i}^{(k)}$ is defined by (1.4).

## Proof

To prove the theorem 1, we replace the multivariable Gimel-function by this multiple integrals contour with the help of (1.1), change the order of integrations which is justified under the conditions mentioned above. Now we evaluate the inner integral with the help of the lemma 1and we interpret the result. Thus we apply the multiple integrals contour (1.1), we obtain the desired result (3.1)

Theorem 2.
$\int_{-\infty}^{\infty} \sin \pi x \beth_{X ; p_{i_{r}}+4, q_{i_{r}}, \tau_{i_{r}}: R_{r}: Y}^{U ; 0, n_{r}: V}\left(\begin{array}{c|c}\mathrm{z}_{1} & \mathbb{A} ; \mathbf{A}, \mathrm{A}_{2}: A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_{r} & \mathbb{B} ; \mathbf{B}: \mathrm{B}\end{array}\right) \mathrm{d} x=$
$\frac{\sin \frac{\pi(b-a)}{2}}{2} \mathrm{I}_{X ; p_{i_{r}}+3, q_{i_{r}}, \tau_{i_{r}}: R_{r}: Y}^{U ; 0, n_{2}: V}\left(\begin{array}{c|c}\mathrm{z}_{1} & \mathbb{A} ; \mathbf{A}, \mathrm{A}_{2}^{\prime}: A \\ \cdot & \cdot \\ \cdot & \mathbb{B} \\ \mathrm{z}_{r} & \mathbb{B} ; \mathbf{B}: \mathrm{B}\end{array}\right)$
where
$A_{2}=\left(a+x ; h_{1}, \cdots, h_{r} ; 1\right),\left(b-x ; h_{1}, \cdots, h_{r} ; 1\right),\left(c+x ; k_{1}, \cdots, k_{r} ; 1\right),\left(d-x ; k_{1}, \cdots, k_{r} ; 1\right)$
and
$A_{2}^{\prime}=\left(\frac{a+b}{2} ; h_{1}, \cdots, h_{r} ; 1\right),\left(\frac{c+d}{2} ; k_{1}, \cdots, k_{r} ; 1\right),\left(a+d-1 ; h_{1}+k_{1}, \cdots, h_{r}+k_{r} ; 1\right)$
provided
$h_{i}, k_{i}>0(i=1, \cdots, r), \operatorname{Re}(a+b+c+d)-2 \sum_{i=1}^{r}\left(h_{i}+k_{i}\right) \min _{1 \leqslant j \leqslant m^{(i)}} \operatorname{Re}\left[D_{j}^{(i)}\left(\frac{d_{j}^{(i)}}{\delta_{j}^{(i)}}\right)\right]>2$.
$\left|\arg \left(z_{k}\right)\right|<\frac{1}{2} A_{i}^{(k)} \pi$ where $A_{i}^{(k)}$ is defined by (1.4).

## Theorem 3.

$\int_{-\infty}^{\infty} \sin \pi x \beth_{X ; p_{i}+2, q_{i}+2, \tau_{i_{r}}: R_{r}: Y}^{U 0, n_{r}: V}\left(\begin{array}{c|c}\mathrm{z}_{1} & \mathbb{A} ; \mathbf{A}, \mathrm{A}_{3}: A \\ \cdot & \cdot \\ \mathrm{z}_{r} & \mathbb{B} ; \mathbf{B}, \mathrm{B}_{3}: B\end{array}\right) \mathrm{d} x=$
$\frac{\sin \frac{\pi(b-a)}{2}}{2} \mathcal{I}_{X ; p_{i}+0, n_{r}: V}^{U}+q_{i_{r}}+1, \tau_{i_{r}}: R_{r}: Y\left(\begin{array}{c|c}\mathrm{z}_{1} & \mathbb{A} ; \mathbf{A}, \mathrm{A}_{3}^{\prime}: A \\ \cdot & \cdot \\ \cdot & { }_{z} \\ \mathrm{z}_{r} & \mathbb{B} ; \mathbf{B}, \mathrm{B}_{3}^{\prime}: B\end{array}\right)$
where
$A_{3}=\left(a+x ; h_{1}, \cdots, h_{r} ; 1\right),\left(b-x ; h_{1}, \cdots, h_{r} ; 1\right) ; B_{3}=\left(1-c-x ; k_{1}, \cdots, k_{r} ; 1\right),\left(1-d+x ; k_{1}, \cdots, k_{r} ; 1\right)$
and
$A_{3}^{\prime}=\left(\frac{a+b}{2} ; h_{1}, \cdots, h_{r} ; 1\right),\left(a+d-1 ; h_{1}-k_{1}, \cdots, h_{r}-k_{r} ; 1\right) ; B_{3}^{\prime}=\left(1-\frac{c+d}{2} ; k_{1}, \cdots, k_{r} ; 1\right)$
provided
$h_{i}, k_{i}, h_{i}-k_{i}>0(i=1, \cdots, r), \operatorname{Re}(a+b+c+d)-2 \sum_{i=1}^{r}\left(h_{i}-k_{i}\right) \min _{1 \leqslant j \leqslant m^{(i)}} \operatorname{Re}\left[D_{j}^{(i)}\left(\frac{d_{j}^{(i)}}{\delta_{j}^{(i)}}\right)\right]>2$.
$\left|\arg \left(z_{k}\right)\right|<\frac{1}{2} A_{i}^{(k)} \pi$ where $A_{i}^{(k)}$ is defined by (1.4).
Theorem 4.
$\int_{-\infty}^{\infty} \sin \pi x \beth_{X ; p_{i_{r}}+2, q_{i_{r}}+2, \tau_{i_{r}}: R_{r}: Y}^{U ; 0, n_{n}: V}\left(\begin{array}{c|c}\mathrm{z}_{1} & \mathbb{A} ; \mathbf{A}, \mathrm{A}_{4}: A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_{r} & \mathbb{B} ; \mathbf{B}, \mathrm{B}_{4}: B\end{array}\right) \mathrm{d} x=$
$\frac{\sin \frac{\pi(b-a)}{2}}{2} \beth_{X ; p_{i_{r}}+1, q_{i_{r}}+2, \tau_{i_{r}}: R_{r}: Y}^{U ; 0, n_{r}: V}\left(\begin{array}{c|c}\mathrm{z}_{1} & \mathbb{A} ; \mathbf{A}, \mathrm{A}_{4}^{\prime}: A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_{r} & \mathbb{B} ; \mathbf{B}, \mathrm{B}_{4}^{\prime}: B\end{array}\right)$
where
$A_{4}=\left(c+x ; h_{1}, \cdots, h_{r} ; 1\right),\left(d-x ; k_{1}, \cdots, k_{r} ; 1\right) ; B_{4}=\left(1-a-x ; h_{1}, \cdots, h_{r} ; 1\right),\left(1-b+x ; h_{1}, \cdots, h_{r} ; 1\right)$
and
$A_{4}^{\prime}=\left(\frac{c+d}{2} ; k_{1}, \cdots, k_{r} ; 1\right) ; B_{4}^{\prime}=\left(1-\frac{a+b}{2} ; h_{1}, \cdots, h_{r} ; 1\right),\left(2-a-d ; h_{1}-k_{1}, \cdots, h_{r}-k_{r} ; 1\right)$
provided
$h_{i}, k_{i}, h_{i}-k_{i}>0(i=1, \cdots, r), \operatorname{Re}(a+b+c+d)+2 \sum_{i=1}^{r}\left(h_{i}-k_{i}\right) \min _{1 \leqslant j \leqslant m^{(i)}} \operatorname{Re}\left[D_{j}^{(i)}\left(\frac{d_{j}^{(i)}}{\delta_{j}^{(i)}}\right)\right]>2$.
$\left|\arg \left(z_{k}\right)\right|<\frac{1}{2} A_{i}^{(k)} \pi$ where $A_{i}^{(k)}$ is defined by (1.4).

## Theorem 5.

$\int_{-\infty}^{\infty} \sin \pi x \beth_{X ; p_{i_{r}}+3, q_{i_{r}}+1, \tau_{i_{r}}: R_{r}: Y}^{U ; 0, n_{r}: V}\left(\begin{array}{c|c}\mathrm{z}_{1} & \mathbb{A} ; \mathbf{A}, \mathrm{A}_{5}: A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_{r} & \mathbb{B} ; \mathbf{B}, \mathrm{B}_{5}: B\end{array}\right) \mathrm{d} x=$
$\frac{\sin \frac{\pi(b-a)}{2}}{2} \mathrm{I}_{X ; p_{i_{r}}+2, q_{i_{r}}, \tau_{i_{r}}: R_{r}: Y}^{U ; 0, n_{r}: V}\left(\begin{array}{c|c}\mathrm{z}_{1} & \mathbb{A} ; \mathbf{A}, \mathrm{A}_{5}^{\prime}: A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_{r} & \mathbb{B} ; \mathbf{B}: \mathrm{B}\end{array}\right)$
where
$A_{5}=\left(a+x ; h_{1}, \cdots, h_{r} ; 1\right),\left(b-x ; h_{1}, \cdots, h_{r} ; 1\right),\left(c+x ; k_{1}, \cdots, k_{r} ; 1\right) ; B_{5}=\left(1-d+x ; k_{1}, \cdots, k_{r} ; 1\right)$
and
$A_{5}^{\prime}=\left(\frac{a+b}{2} ; h_{1}, \cdots, h_{r} ; 1\right),\left(a+d-1 ; h_{1}-k_{1} ; \cdots ; h_{r}-k_{r} ; 1\right)$
provided
$h_{i}, k_{i}>0(i=1, \cdots, r), \operatorname{Re}(a+b+c+d)-2 \sum_{i=1}^{r} h_{i} \min _{1 \leqslant j \leqslant m^{(i)}} \operatorname{Re}\left[D_{j}^{(i)}\left(\frac{d_{j}^{(i)}}{\delta_{j}^{(i)}}\right)\right]>2$.
$\left|\arg \left(z_{k}\right)\right|<\frac{1}{2} A_{i}^{(k)} \pi$ where $A_{i}^{(k)}$ is defined by (1.4).
To prove the theorems 2 to 5 , we use the similar methods that theorem 1 .

Remark : We can establish another five integrals which involves $\cos \pi x$ in place of $\sin \pi x$ and using the lemma 2.

## 4. Conclusion.

The Gimel-function of several variables presented in this paper, are quite basic in nature. Therefore, on specializing the parameters of this function, we may obtain various known and (news) integrals with respect to parameters concerning the special functions of one variable and several variables. For example Goyal and Agrawal [5] have studied these integrals about the I-function of two variables defined by these authors mentioned above [4].

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