

Integration of certain generalized Gimel-function with respect to their parameters

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ABSTRACT

The object of the present paper is to establish a new class of two-dimensional series expansion of generalized multivariable Gimel-function involving Bessel's function.

KEYWORDS : Generalized multivariable Gimel-function, multiple integral contours, Two-dimensional series, Bessel's function, orthogonal property.

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1. Introduction and preliminaries.

Throughout this paper, let \mathbb{C}, \mathbb{R} and \mathbb{N} be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The objective of this paper is to introduce a new class of two dimensional series expansions for product of generalized multivariable Gimel-function defined here and represents two dimensional series expansion of this class. We define a generalized transcendental function of several complex variables noted \mathfrak{J} .

$$\mathfrak{J}(z_1, \dots, z_r) = \mathfrak{J}_{\substack{m_2, n_2; m_3, n_3; \dots; m_r, n_r; m^{(1)}, n^{(1)}; m^{(2)}, n^{(2)}; \dots; m^{(r)}, n^{(r)} \\ p_{i_2}, q_{i_2}, \tau_{i_2}; R_2; p_{i_3}, q_{i_3}, \tau_{i_3}; R_3; \dots; p_{i_r}, q_{i_r}, \tau_{i_r}; R_r; p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}; \dots; p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)}}} \left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{matrix} \right)$$

$$[(a_{2j}; \alpha_{2j}^{(1)}, \alpha_{2j}^{(2)}; A_{2j})]_{1, n_2}, [\tau_{i_2}(a_{2ji_2}; \alpha_{2ji_2}^{(1)}, \alpha_{2ji_2}^{(2)}; A_{2ji_2})]_{n_2+1, p_{i_2}}, [(a_{3j}; \alpha_{3j}^{(1)}, \alpha_{3j}^{(2)}, \alpha_{3j}^{(3)}; A_{3j})]_{1, n_3},$$

$$[(b_{2j}; \beta_{2j}^{(1)}, \beta_{2j}^{(2)}; B_{2j})]_{1, m_2}, [\tau_{i_2}(b_{2ji_2}; \beta_{2ji_2}^{(1)}, \beta_{2ji_2}^{(2)}; B_{2ji_2})]_{m_2+1, q_{i_2}}, [(b_{3j}; \beta_{3j}^{(1)}, \beta_{3j}^{(2)}, \beta_{3j}^{(3)}; B_{3j})]_{1, m_3},$$

$$[\tau_{i_3}(a_{3ji_3}; \alpha_{3ji_3}^{(1)}, \alpha_{3ji_3}^{(2)}, \alpha_{3ji_3}^{(3)}; A_{3ji_3})]_{n_3+1, p_{i_3}}; \dots; [(a_{rj}; \alpha_{rj}^{(1)}, \dots, \alpha_{rj}^{(r)}; A_{rj})]_{1, n_r},$$

$$[\tau_{i_3}(b_{3ji_3}; \beta_{3ji_3}^{(1)}, \beta_{3ji_3}^{(2)}, \beta_{3ji_3}^{(3)}; B_{3ji_3})]_{m_3+1, q_{i_3}}; \dots; [(b_{rj}; \beta_{rj}^{(1)}, \dots, \beta_{rj}^{(r)}; B_{rj})]_{1, m_r},$$

$$[\tau_{i_r}(a_{rji_r}; \alpha_{rji_r}^{(1)}, \dots, \alpha_{rji_r}^{(r)}; A_{rji_r})]_{n_r+1, p_{i_r}} : [(c_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})]_{1, n^{(1)}}, [\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}, \gamma_{ji^{(1)}}^{(1)}; C_{ji^{(1)}}^{(1)})]_{n^{(1)}+1, p_{i^{(1)}}}]$$

$$[\tau_{i_r}(b_{rji_r}; \beta_{rji_r}^{(1)}, \dots, \beta_{rji_r}^{(r)}; B_{rji_r})]_{m_r+1, q_{i_r}} : [(d_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})]_{1, m^{(1)}}, [\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}, \delta_{ji^{(1)}}^{(1)}; D_{ji^{(1)}}^{(1)})]_{m^{(1)}+1, q_{i^{(1)}}}]$$

$$; \dots; [(c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})]_{1, n^{(r)}}, [\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}, \gamma_{ji^{(r)}}^{(r)}; C_{ji^{(r)}}^{(r)})]_{n^{(r)}+1, p_{i^{(r)}}}]$$

$$; \dots; [(d_j^{(r)}, \delta_j^{(r)}; D_j^{(r)})]_{1, m^{(r)}}, [\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}, \delta_{ji^{(r)}}^{(r)}; D_{ji^{(r)}}^{(r)})]_{m^{(r)}+1, q_{i^{(r)}}}]$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \psi(s_1, \dots, s_r) \prod_{k=1}^r \theta_k(s_k) z_k^{s_k} ds_1 \dots ds_r \tag{1.1}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1, \dots, s_r) = \frac{\prod_{j=1}^{m_2} \Gamma^{B_{2j}}(b_{2j} - \sum_{k=1}^2 \beta_{2j}^{(k)} s_k) \prod_{j=1}^{n_2} \Gamma^{A_{2j}}(1 - a_{2j} + \sum_{k=1}^2 \alpha_{2j}^{(k)} s_k)}{\sum_{i_2=1}^{R_2} [\tau_{i_2} \prod_{j=n_2+1}^{p_{i_2}} \Gamma^{A_{2ji_2}}(a_{2ji_2} - \sum_{k=1}^2 \alpha_{2ji_2}^{(k)} s_k) \prod_{j=m_2+1}^{q_{i_2}} \Gamma^{B_{2ji_2}}(1 - b_{2ji_2} + \sum_{k=1}^2 \beta_{2ji_2}^{(k)} s_k)]}$$

$$\frac{\prod_{j=1}^{m_3} \Gamma^{B_{3j}}(b_{3j} - \sum_{k=1}^3 \beta_{3j}^{(k)} s_k) \prod_{j=1}^{n_3} \Gamma^{A_{3j}}(1 - a_{3j} + \sum_{k=1}^3 \alpha_{3j}^{(k)} s_k)}{\sum_{i_3=1}^{R_3} [\tau_{i_3} \prod_{j=n_3+1}^{p_{i_3}} \Gamma^{A_{3ji_3}}(a_{3ji_3} - \sum_{k=1}^3 \alpha_{3ji_3}^{(k)} s_k) \prod_{j=m_3+1}^{q_{i_3}} \Gamma^{B_{3ji_3}}(1 - b_{3ji_3} + \sum_{k=1}^3 \beta_{3ji_3}^{(k)} s_k)]}$$

$$\frac{\prod_{j=1}^{m_r} \Gamma^{B_{rj}}(b_{rj} - \sum_{k=1}^r \beta_{rj}^{(k)} s_k) \prod_{j=1}^{n_r} \Gamma^{A_{rj}}(1 - a_{rj} + \sum_{k=1}^r \alpha_{rj}^{(k)} s_k)}{\sum_{i_r=1}^{R_r} [\tau_{i_r} \prod_{j=n_r+1}^{p_{i_r}} \Gamma^{A_{rji_r}}(a_{rji_r} - \sum_{k=1}^r \alpha_{rji_r}^{(k)} s_k) \prod_{j=1}^{q_{i_r}} \Gamma^{B_{rji_r}}(1 - b_{rji_r} + \sum_{k=1}^r \beta_{rji_r}^{(k)} s_k)]}$$
(1.2)

and

$$\theta_k(s_k) = \frac{\prod_{j=1}^{m^{(k)}} \Gamma^{D_j^{(k)}}(d_j^{(k)} - \delta_j^{(k)} s_k) \prod_{j=1}^{n^{(k)}} \Gamma^{C_j^{(k)}}(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}} [\tau_{i^{(k)}} \prod_{j=m^{(k)}+1}^{q_{i^{(k)}}} \Gamma^{D_{ji^{(k)}}^{(k)}}(1 - d_{ji^{(k)}}^{(k)} + \delta_{ji^{(k)}}^{(k)} s_k) \prod_{j=n^{(k)}+1}^{p_{i^{(k)}}} \Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)} - \gamma_{ji^{(k)}}^{(k)} s_k)]}$$
(1.3)

1) $[(c_j^{(1)}; \gamma_j^{(1)})]_{1, n_1}$ stands for $(c_1^{(1)}; \gamma_1^{(1)}), \dots, (c_{n_1}^{(1)}; \gamma_{n_1}^{(1)})$.

2) $m_2, n_2, \dots, m_r, n_r, m^{(1)}, n^{(1)}, \dots, m^{(r)}, n^{(r)}, p_{i_2}, q_{i_2}, R_2, \tau_{i_2}, \dots, p_{i_r}, q_{i_r}, R_r, \tau_{i_r}, p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}, R^{(r)} \in \mathbb{N}$ and verify :

$$0 \leq m_2 \leq q_{i_2}, 0 \leq n_2 \leq p_{i_2}, \dots, 0 \leq m_r \leq q_{i_r}, 0 \leq n_r \leq p_{i_r}, 0 \leq m^{(1)} \leq q_{i^{(1)}}, \dots, 0 \leq m^{(r)} \leq q_{i^{(r)}}.$$

3) $\tau_{i_2} (i_2 = 1, \dots, R_2) \in \mathbb{R}^+$; $\tau_{i_r} \in \mathbb{R}^+ (i_r = 1, \dots, R_r)$; $\tau_{i^{(k)}} \in \mathbb{R}^+ (i = 1, \dots, R^{(k)}), (k = 1, \dots, r)$.

4) $\gamma_j^{(k)}, C_j^{(k)} \in \mathbb{R}^+; (j = 1, \dots, n_k); (k = 1, \dots, r); \delta_j^{(k)}, D_j^{(k)} \in \mathbb{R}^+; (j = 1, \dots, m_k); (k = 1, \dots, r)$.

$\alpha_{kj}^{(l)}, A_{kj} \in \mathbb{R}^+; (j = 1, \dots, n_k); (k = 2, \dots, r); (l = 1, \dots, k)$.

$\beta_{kj}^{(l)}, B_{kj} \in \mathbb{R}^+; (j = 1, \dots, m_k); (k = 2, \dots, r); (l = 1, \dots, k)$.

$\alpha_{kji_k}^{(l)}, A_{kji_k} \in \mathbb{R}^+; (j = n_k + 1, \dots, p_{i_k}); (k = 2, \dots, r); (l = 1, \dots, k)$.

$\beta_{kji_k}^{(l)}, B_{kji_k} \in \mathbb{R}^+; (j = m_k + 1, \dots, q_{i_k}); (k = 2, \dots, r); (l = 1, \dots, k)$.

$\delta_{ji^{(k)}}^{(k)} \in \mathbb{R}^+; (i = 1, \dots, R^{(k)}); (j = m_k + 1, \dots, q_{i^{(k)}}); (k = 1, \dots, r)$.

$\gamma_{ji^{(k)}}^{(k)} \in \mathbb{R}^+; (i = 1, \dots, R^{(k)}); (j = n_k + 1, \dots, p_{i^{(k)}}); (k = 1, \dots, r)$.

5) $c_j^{(k)} \in \mathbb{C}; (j = 1, \dots, n_k); (k = 1, \dots, r); d_j^{(k)} \in \mathbb{C}; (j = 1, \dots, m_k); (k = 1, \dots, r)$.

$a_{kji_k} \in \mathbb{C}; (j = n_k + 1, \dots, p_{i_k}); (k = 2, \dots, r)$.

$b_{kji_k} \in \mathbb{C}; (j = m_k + 1, \dots, q_{i_k}); (k = 2, \dots, r)$.

$d_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \dots, R^{(k)}); (j = m_k + 1, \dots, q_{i^{(k)}}); (k = 1, \dots, r)$.

$$\gamma_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \dots, R^{(k)}); (j = n_k + 1, \dots, p_{i^{(k)}}); (k = 1, \dots, r).$$

The contour L_k is in the $s_k (k = 1, \dots, r)$ - plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}} \left(1 - a_{2j} + \sum_{k=1}^2 \alpha_{2j}^{(k)} s_k \right) (j = 1, \dots, n_2), \Gamma^{A_{3j}} \left(1 - a_{3j} + \sum_{k=1}^3 \alpha_{3j}^{(k)} s_k \right) (j = 1, \dots, n_3), \dots, \Gamma^{A_{rj}} \left(1 - a_{rj} + \sum_{i=1}^r \alpha_{rj}^{(i)} \right) (j = 1, \dots, n_r), \Gamma^{C_j^{(k)}} \left(1 - c_j^{(k)} + \gamma_j^{(k)} s_k \right) (j = 1, \dots, n^{(k)}) (k = 1, \dots, r)$ to the right of the contour L_k and the poles of $\Gamma^{B_{2j}} \left(b_{2j} - \sum_{k=1}^2 \beta_{2j}^{(k)} s_k \right) (j = 1, \dots, m_2), \Gamma^{B_{3j}} \left(b_{3j} - \sum_{k=1}^3 \beta_{3j}^{(k)} s_k \right) (j = 1, \dots, m_3), \dots, \Gamma^{B_{rj}} \left(b_{rj} - \sum_{i=1}^r \beta_{rj}^{(i)} \right) (j = 1, \dots, m_r), \Gamma^{D_j^{(k)}} \left(d_j^{(k)} - \delta_j^{(k)} s_k \right) (j = 1, \dots, m^{(k)}) (k = 1, \dots, r)$ lie to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$$|arg(z_k)| < \frac{1}{2} A_i^{(k)} \pi \text{ where}$$

$$A_i^{(k)} = \sum_{j=1}^{m^{(k)}} D_j^{(k)} \delta_j^{(k)} + \sum_{j=1}^{n^{(k)}} C_j^{(k)} \gamma_j^{(k)} - \tau_{i^{(k)}} \left(\sum_{j=m^{(k)}+1}^{q_{i^{(k)}}} D_{ji^{(k)}}^{(k)} \delta_{ji^{(k)}}^{(k)} + \sum_{j=n^{(k)}+1}^{p_{i^{(k)}}} C_{ji^{(k)}}^{(k)} \gamma_{ji^{(k)}}^{(k)} \right) + \sum_{j=1}^{n_2} A_{2j} \alpha_{2j}^{(k)} + \sum_{j=1}^{m_2} B_{2j} \beta_{2j}^{(k)} - \tau_{i_2} \left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2} \alpha_{2ji_2}^{(k)} + \sum_{j=m_2+1}^{q_{i_2}} B_{2ji_2} \beta_{2ji_2}^{(k)} \right) + \dots + \sum_{j=1}^{n_r} A_{rj} \alpha_{rj}^{(k)} + \sum_{j=1}^{m_r} B_{rj} \beta_{rj}^{(k)} - \tau_{i_r} \left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r} \alpha_{rji_r}^{(k)} + \sum_{j=m_r+1}^{q_{i_r}} B_{rji_r} \beta_{rji_r}^{(k)} \right) \tag{1.4}$$

Following the lines of Braaksma ([3] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1, \dots, z_r) = O(|z_1|^{\alpha_1}, \dots, |z_r|^{\alpha_r}), \max(|z_1|, \dots, |z_r|) \rightarrow 0$$

$$\aleph(z_1, \dots, z_r) = O(|z_1|^{\beta_1}, \dots, |z_r|^{\beta_r}), \min(|z_1|, \dots, |z_r|) \rightarrow \infty \text{ where } i = 1, \dots, r :$$

$$\alpha_i = \min_{\substack{1 \leq j \leq m_i \\ 1 \leq j \leq m^{(i)}}} \operatorname{Re} \left(\sum_{h=2}^r \sum_{h'=1}^h B_{hj} \frac{b_{hj}}{\beta_{hj}^{h'}} + D_j^{(i)} \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \text{ and } \beta_i = \max_{\substack{1 \leq j \leq n_i \\ 1 \leq j \leq n^{(i)}}} \operatorname{Re} \left(\sum_{h=2}^r \sum_{h'=1}^h A_{hj} \frac{a_{hj} - 1}{\alpha_{hj}^{h'}} + C_j^{(i)} \frac{c_j^{(i)} - 1}{\gamma_j^{(i)}} \right)$$

Remark 1.

If $m_2 = n_2 = \dots = m_{r-1} = n_{r-1} = p_{i_2} = q_{i_2} = \dots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \dots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the generalized multivariable Gimel-function reduces in the generalized multivariable Aleph- function (extension of multivariable Aleph-function defined by Ayant [2]).

Remark 2.

If $m_2 = n_2 = \dots = m_r = n_r = p_{i_2} = q_{i_2} = \dots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \dots = \tau_{i_r} = \tau_{i_2(1)} = \dots = \tau_{i_r(r)} = R_2 = \dots = R_r = R^{(1)} = \dots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in a generalized multivariable I-function (extension of multivariable I-function defined by Prathima et al. [7]).

Remark 3.

If $A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \dots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$ and $\tau_{i_2} = \dots = \tau_{i_r} = \tau_{i_2(1)} = \dots = \tau_{i_r(r)} = R_2 = \dots = R_r = R^{(1)} = \dots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in generalized of multivariable I-function (extension of multivariable I-function defined by Prasad [6]).

Remark 4.

If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the generalized multivariable H-function (extension of multivariable H-function defined by Srivastava and Panda [9,10]).

In your investigation, we shall use the following notations.

$$\mathbb{A} = [(a_{2j}; \alpha_{2j}^{(1)}, \alpha_{2j}^{(2)}; A_{2j})]_{1, n_2}, [\tau_{i_2}(a_{2ji_2}; \alpha_{2ji_2}^{(1)}, \alpha_{2ji_2}^{(2)}; A_{2ji_2})]_{n_2+1, p_{i_2}}, [(a_{3j}; \alpha_{3j}^{(1)}, \alpha_{3j}^{(2)}, \alpha_{3j}^{(3)}; A_{3j})]_{1, n_3},$$

$$[\tau_{i_3}(a_{3ji_3}; \alpha_{3ji_3}^{(1)}, \alpha_{3ji_3}^{(2)}, \alpha_{3ji_3}^{(3)}; A_{3ji_3})]_{n_3+1, p_{i_3}}; \dots; [(a_{(r-1)j}; \alpha_{(r-1)j}^{(1)}, \dots, \alpha_{(r-1)j}^{(r-1)}; A_{(r-1)j})]_{1, n_{r-1}},$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}}; \alpha_{(r-1)ji_{r-1}}^{(1)}, \dots, \alpha_{(r-1)ji_{r-1}}^{(r-1)}; A_{(r-1)ji_{r-1}})]_{n_{r-1}+1, p_{i_{r-1}}} \tag{1.5}$$

$$\mathbf{A} = [(a_{rj}; \alpha_{rj}^{(1)}, \dots, \alpha_{rj}^{(r)}; A_{rj})]_{1, n_r}, [\tau_{i_r}(a_{rji_r}; \alpha_{rji_r}^{(1)}, \dots, \alpha_{rji_r}^{(r)}; A_{rji_r})]_{n+1, p_{i_r}} \tag{1.6}$$

$$A = [(c_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})]_{1, n^{(1)}}, [\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}, \gamma_{ji^{(1)}}^{(1)}; C_{ji^{(1)}}^{(1)})]_{n^{(1)}+1, p_i^{(1)}}; \dots;$$

$$[(c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})]_{1, n^{(r)}}, [\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}, \gamma_{ji^{(r)}}^{(r)}; C_{ji^{(r)}}^{(r)})]_{n^{(r)}+1, p_i^{(r)}} \tag{1.7}$$

$$\mathbb{B} = [(b_{2j}; \beta_{2j}^{(1)}, \beta_{2j}^{(2)}; B_{2j})]_{1, m_2}, [\tau_{i_2}(b_{2ji_2}; \beta_{2ji_2}^{(1)}, \beta_{2ji_2}^{(2)}; B_{2ji_2})]_{m_2+1, q_{i_2}}, [(b_{3j}; \beta_{3j}^{(1)}, \beta_{3j}^{(2)}, \beta_{3j}^{(3)}; B_{3j})]_{1, m_3},$$

$$[\tau_{i_3}(b_{3ji_3}; \beta_{3ji_3}^{(1)}, \beta_{3ji_3}^{(2)}, \beta_{3ji_3}^{(3)}; B_{3ji_3})]_{m_3+1, q_{i_3}}; \dots; [(b_{(r-1)j}; \beta_{(r-1)j}^{(1)}, \dots, \beta_{(r-1)j}^{(r-1)}; B_{(r-1)j})]_{1, m_{r-1}},$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}}; \beta_{(r-1)ji_{r-1}}^{(1)}, \dots, \beta_{(r-1)ji_{r-1}}^{(r-1)}; B_{(r-1)ji_{r-1}})]_{m_{r-1}+1, q_{i_{r-1}}} \tag{1.8}$$

$$\mathbf{B} = [(b_{rj}; \beta_{rj}^{(1)}, \dots, \beta_{rj}^{(r)}; B_{rj})]_{1, m_r}, [\tau_{i_r}(b_{rji_r}; \beta_{rji_r}^{(1)}, \dots, \beta_{rji_r}^{(r)}; B_{rji_r})]_{m_r+1, q_{i_r}} \tag{1.9}$$

$$B = [(d_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})]_{1, m^{(1)}}, [\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}, \delta_{ji^{(1)}}^{(1)}; D_{ji^{(1)}}^{(1)})]_{m^{(1)}+1, q_i^{(1)}}; \dots;$$

$$[(d_j^{(r)}, \delta_j^{(r)}; D_j^{(r)})]_{1, m^{(r)}}, [\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}, \delta_{ji^{(r)}}^{(r)}; D_{ji^{(r)}}^{(r)})]_{m^{(r)}+1, q_i^{(r)}} \tag{1.10}$$

$$U = m_2, n_2; m_3, n_3; \dots; m_{r-1}, n_{r-1}; V = m^{(1)}, n^{(1)}; m^{(2)}, n^{(2)}; \dots; m^{(r)}, n^{(r)} \tag{1.11}$$

$$X = p_{i_2}, q_{i_2}, \tau_{i_2}; R_2; \dots; p_{i_{r-1}}, q_{i_{r-1}}, \tau_{i_{r-1}}; R_{r-1}; Y = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}; \dots; p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{1.12}$$

2. Required results.

The following results will be used throughout the paper. Luke [5] gives the formulae.

Lemma 1.

$$\int_0^\infty x^{\rho-1} J_u(x) dx = \frac{2^{\rho-1} \Gamma(\frac{u+\rho}{2})}{\Gamma(1 - \frac{\rho-u}{2})} \tag{2.1}$$

provided $Re(\rho + u) > 0, Re(\rho) < 0$.

Lemma 2.

$$\int_0^\infty y^{\sigma-1} \cos y J_\nu(x) dy = \frac{2^{\sigma-1} \sqrt{\pi} \Gamma(\frac{1}{2} - \sigma) \Gamma(\frac{\nu+\sigma}{2})}{\Gamma(1 - \frac{\nu-\sigma}{2}) \Gamma(\frac{1-\nu-\sigma}{2}) \Gamma(\frac{1+\nu-\sigma}{2})} \tag{2.2}$$

with $Re(\sigma + \nu) > 0$.

Orthogonality property for Bessel's function [8]

Lemma 3.

$$\int_0^\infty x^{-1} J_{u+2m+1}(x) J_{u+2n+1}(x) dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{(u+2n+1)^{-1}}{2}, & \text{if } m = n \end{cases} \tag{2.3}$$

provided $Re(u) + m + n > -1$.

3. Main integrals.

We shall evaluate the integrals that will be used in the expansions.

Theorem 1.

$$\int_0^\infty x^{\rho-1} J_u(x) \mathfrak{J} \left(z_1 x^{2h_1}, \dots, z_r x^{2h_r} \right) dx = 2^{\rho-1} \mathfrak{J}_{X;p_{i_r}+2, q_{i_r}, \tau_{i_r}; R_r; Y}^{U; m_r, n_r+1; V} \left(\begin{matrix} z_1 4^{h_1} \\ \vdots \\ z_r 4^{h_r} \end{matrix} \middle| \begin{matrix} \mathbb{A}; (1 - \frac{\rho+u}{2}; h_1, \dots, h_r; 1), \mathbf{A}, (1 - \frac{\rho-u}{2}; h_1, \dots, h_r; 1) : A \\ \vdots \\ \mathbb{B}; \mathbf{B} : B \end{matrix} \right) \tag{3.1}$$

provided

$$h_i > 0 (i = 1, \dots, r), Re(\rho + u) + 2 \sum_{i=1}^r h_i \min_{\substack{1 \leq j \leq m_i \\ 1 \leq j \leq m^{(i)}}} Re \left(\sum_{h=2}^r \sum_{h'=1}^h B_{hj} \frac{b_{hj}}{\beta_{hj}^{h'}} + D_j^{(i)} \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) > 0.$$

$$Re(\rho) + 2 \sum_{i=1}^r h_i \max_{\substack{1 \leq j \leq n_i \\ 1 \leq j \leq n^{(i)}}} Re \left(\sum_{h=2}^r \sum_{h'=1}^h A_{hj} \frac{a_{hj} - 1}{\alpha_{hj}^{h'}} + C_j^{(i)} \frac{c_j^{(i)} - 1}{\gamma_j^{(i)}} \right) < 0.$$

$$|arg(4^{h_k} z_k)| < \frac{1}{2} A_i^{(k)} \pi \text{ where } A_i^{(k)} \text{ is defined by (1.4).}$$

Theorem 2.

$$\int_0^\infty x^{\sigma-1} \cos x J_u(x) \mathfrak{J} \left(z_1 x^{2k_1}, \dots, z_r x^{2k_r} \right) dx = 2^{\sigma-1} \sqrt{\pi} \mathfrak{J}_{X;p_{i_r}+3, q_{i_r}+2, \tau_{i_r}; R_r; Y}^{U; m_r+1, n_r+1; V} \left(\begin{matrix} z_1 4^{h_1} \\ \vdots \\ z_r 4^{h_r} \end{matrix} \middle| \begin{matrix} \mathbb{A}; (1 - \frac{\sigma+u}{2}; k_1, \dots, k_r; 1), \mathbf{A}, (\frac{1-\sigma-u}{2}; k_1, \dots, k_r; 1), (\frac{1-\sigma+u}{2}; k_1, \dots, k_r; 1) : A \\ \vdots \\ \mathbb{B}; (\frac{1}{2} - \sigma; 2k_1, \dots, 2k_r; 1), \mathbf{B}, (\frac{u-\sigma}{2}; k_1, \dots, k_r; 1) : B \end{matrix} \right) \tag{3.2}$$

provided

$$k_i > 0 (i = 1, \dots, r), Re(\sigma + u) + 2 \sum_{i=1}^r k_i \min_{\substack{1 \leq j \leq m_i \\ 1 \leq j \leq m^{(i)}}} Re \left(\sum_{h=2}^r \sum_{h'=1}^h B_{hj} \frac{b_{hj}}{\beta_{hj}^{h'}} + D_j^{(i)} \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) > 0.$$

$$|arg(4^{k_i} z_i)| < \frac{1}{2} A_i^{(k)} \pi \text{ where } A_i^{(k)} \text{ is defined by (1.4).}$$

Proof

To establish (3.1), express generalized multivariable Gimel-function as in (1.1), change the order of integrations. Evaluate the inner integral with the help of lemma 1 and applying again (1.1), we get the desired result (3.1). Apply the similar methods to prove (3.2) with the help of lemma 2 instead to lemma 1.

4. Two dimensional series expansion.

The two dimensional series expansion to be established is

Theorem 3.

$$x^\rho y^\sigma \cos y \mathfrak{J} (z_1 x^{2h_1} y^{2k_1}, \dots, z_r x^{2h_r} y^{2k_r}) = 2^{\rho+\sigma} \sqrt{\pi}$$

$$\sum_{s,t=0}^{\infty} (v + 2s + 1)(u + 2t + 1) J_{v+2s+1}(x) J_{u+2t+1}(y)$$

$$\mathfrak{J}_{X; p_{i_r} + \delta, q_{i_r} + 2, \tau_{i_r}; R_r; Y}^{U; m_r + 1, n_r + 2; V} \left(\begin{array}{c|c} z_1 4^{h_1} & \mathbb{A}; \mathbf{A}_1, \mathbf{A}, A_2 : A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r 4^{h_r} & \mathbb{B}; \mathbf{B}_1, \mathbf{B}, B_2 : B \end{array} \right) \tag{4.1}$$

where

$$A_1 = \left(\frac{1 - \rho - v - 2s}{2}; h_1, \dots, h_r; 1 \right), \left(\frac{1 - \sigma - u - 2t}{2}; k_1, \dots, k_r; 1 \right) \tag{4.2}$$

$$A_2 = \left(\frac{3 + v - \rho + 2s}{2}; h_1, \dots, h_r; 1 \right), \left(\frac{2 + u - \sigma + 2t}{2}; k_1, \dots, k_r; 1 \right), \left(-\frac{1 + u + 2t - \sigma}{2}; k_1, \dots, k_r; 1 \right) \tag{4.3}$$

$$B_1 = \left(\frac{1}{2} - \sigma; 2k_1, \dots, 2k_r; 1 \right); B_2 = \left(\frac{1 + u + 2t - \sigma}{2}; k_1, \dots, k_r; 1 \right) \tag{4.4}$$

valid under the existence conditions mentioned in (3.1) and (3.2).

Proof

To establish the above two dimensional series expansion, let

$$x^\rho y^\sigma \cos y \mathfrak{J} (z_1 x^{2h_1} y^{2k_1}, \dots, z_r x^{2h_r} y^{2k_r}) = \sum_{s,t=0}^{\infty} C_{s,t} J_{v+2s+1}(x) J_{u+2t+1}(y) \tag{4.5}$$

Multiplying both sides of equation (4.5) by $y^{-1} J_{u+2v'-1}(y)$ and integrating it with respect to y from 0 to ∞ , using the theorem 2 and lemma 3.

Multiplying both sides of equation (4.5) by $x^{-1} J_{v+2u'-1}(x)$ and integrating it with respect to x from 0 to ∞ , using the theorem 1 and lemma 3, we obtain the value of $C_{s,t}$.

Finally substituting the value of $C_{s,t}$ in (4.5), we obtain the desired result (4.1).

Remarks :

On applying the same procedure as above, we can establish several other forms of two dimensional expansions of this class with the help of alternative forms of (3.1) and (3.2).

Agarwal et al. [1] have obtained the same results about I-function of two variables defined by Goyal and Agarwal [4].

5. Conclusion.

Since on specializing the parameters of generalized Gimel-function of several variables yield almost all special functions appearing in Applied Mathematics and Physical Sciences. Therefore the result presented in this study is of a general character and hence may encompass several cases of interest.

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