# Unified integrals about certain Gimel-function

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ABSTRACT
The object of the present paper is to obtain unified integrals involving the multivariable Gimel-function and a general polynomial having general arguments.

KEYWORDS : Multivariable Gimel-function, multiple integral contours, general .polynomial, unified integral.

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## 1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C}, \mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

In this paper two finite integrals involving the generalized multivariable function noted $\gimel$ presented here and a general polynomial having arguments of the type $x^l(b-x)^m(x^t+a^t)^{-\lambda}$ have been evaluated. These integrals, besides being of very general character have been put in a compact form avoiding the occurrence of infinite series and thus making them useful in applications. Our findings provide interesting unifications and extensions of a number of results. For the sake of illustration, four new integrals two of which involve the Konhauser biorthogonal, Hermite and Laguerre polynomials have been obtained as special cases of the second integral.

We define a generalized transcendental function of several complex variables noted $\gimel$.

$$\gimel(z_1,\cdots,z_r) = \gimel^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array}\right|$$

$$[(a_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}};[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$
$$[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}};$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(a_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;\quad \ldots$$

$$[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{n_r+1,p_r}]:\ [(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}]$$
$$[\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{1,q_r}]:\ [(d_j^{(1)}),\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}]$$

$$\left.\begin{array}{l};\cdots;[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,n^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_j^{(r)})_{n^{(r)}+1,p_i^{(r)}}] \\ ;\cdots;[(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_j^{(r)})_{m^{(r)}+1,q_i^{(r)}}]\end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha_{2ji_2}^{(k)}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta_{2ji2}^{(k)}s_k)]}$$

$$\frac{\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \quad (1.2)$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \quad (1.3)$$

1) $[(c_j^{(1)};\gamma_j^{(1)}]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $n_2,\cdots,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2,0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant m_r,0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m_k);(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n_k);(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m_k);(k=1,\cdots,r)$.

$a_{kji_k}\in\mathbb{C};(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r)$.

$b_{kji_k}\in\mathbb{C};(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r)$.

$d_{ji^{(k)}}^{(k)}\in\mathbb{C};(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r).$

The contour $L_k$ is in the $s_k(k = 1, \cdots, r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where $\sigma$ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1 - a_{2j} + \sum_{k=1}^{2} \alpha_{2j}^{(k)} s_k\right) (j = 1, \cdots, n_2), \Gamma^{A_{3j}}\left(1 - a_{3j} + \sum_{k=1}^{3} \alpha_{3j}^{(k)} s_k\right)$ $(j = 1, \cdots, n_3), \cdots, \Gamma^{A_{rj}}\left(1 - a_{rj} + \sum_{i=1}^{r} \alpha_{rj}^{(i)}\right) (j = 1, \cdots, n_r)$, $\Gamma^{C_j^{(k)}}\left(1 - c_j^{(k)} + \gamma_j^{(k)} s_k\right) (j = 1, \cdots, n^{(k)})(k = 1, \cdots, r)$to the right of the contour $L_k$ and the poles of $\Gamma^{D_j^{(k)}}\left(d_j^{(k)} - \delta_j^{(k)} s_k\right) (j = 1, \cdots, m^{(k)})(k = 1, \cdots, r)$ lie to the left of the contour $L_k$. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2} A_i^{(k)} \pi$ where

$$A_i^{(k)} = \sum_{j=1}^{m^{(k)}} D_j^{(k)} \delta_j^{(k)} + \sum_{j=1}^{n^{(k)}} C_j^{(k)} \gamma_j^{(k)} - \tau_{i^{(k)}} \left( \sum_{j=m^{(k)}+1}^{q_i^{(k)}} D_{ji^{(k)}}^{(k)} \delta_{ji^{(k)}}^{(k)} + \sum_{j=n^{(k)}+1}^{p_i^{(k)}} C_{ji^{(k)}}^{(k)} \gamma_{ji^{(k)}}^{(k)} \right)$$

$$-\tau_{i_2} \left( \sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2} \alpha_{2ji_2}^{(k)} + \sum_{j=1}^{q_{i_2}} B_{2ji_2} \beta_{2ji_2}^{(k)} \right) - \cdots - \tau_{i_r} \left( \sum_{j=n_r+1}^{p_{i_r}} A_{rji_r} \alpha_{rji_r}^{(k)} + \sum_{j=1}^{q_{i_r}} B_{rji_r} \beta_{rji_r}^{(k)} \right) \tag{1.4}$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1, \cdots, z_r) = 0( |z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r} ), max( |z_1|, \cdots, |z_r| ) \rightarrow 0$$

$$\aleph(z_1, \cdots, z_r) = 0( |z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r} ), min( |z_1|, \cdots, |z_r| ) \rightarrow \infty \text{ where } i = 1, \cdots, r :$$

$$\alpha_i = \min_{1 \leqslant j \leqslant m^{(i)}} Re \left[ D_j^{(i)} \left( \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] \text{ and } \beta_i = \max_{1 \leqslant j \leqslant n^{(i)}} Re \left[ C_j^{(i)} \left( \frac{c_j^{(i)} - 1}{\gamma_j^{(i)}} \right) \right]$$

**Remark 1.**
If $n_2 = \cdots = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$ $A_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the multivariable Gimel-function reduces in the multivariable Aleph- function defined by Ayant [1].

**Remark 2.**
If $n_2 = \cdots = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [5].

**Remark 3.**
If$A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [4].

**Remark 4.**
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H-function defined by Srivastava and Panda [8,9].

In your investigation, we shall use the following notations.

$\mathbb{A} = [(\mathrm{a}_{2j}; \alpha_{2j}^{(1)}, \alpha_{2j}^{(2)}; A_{2j})]_{1,n_2}, [\tau_{i_2}(a_{2ji_2}; \alpha_{2ji_2}^{(1)}, \alpha_{2ji_2}^{(2)}; A_{2ji_2})]_{n_2+1,p_{i_2}}, [(a_{3j}; \alpha_{3j}^{(1)}, \alpha_{3j}^{(2)}, \alpha_{3j}^{(3)}; A_{3j})]_{1,n_3},$

$[\tau_{i_3}(a_{3ji_3}; \alpha_{3ji_3}^{(1)}, \alpha_{3ji_3}^{(2)}, \alpha_{3ji_3}^{(3)}; A_{3ji_3})]_{n_3+1,p_{i_3}}; \cdots; [(\mathrm{a}_{(r-1)j}; \alpha_{(r-1)j}^{(1)}, \cdots, \alpha_{(r-1)j}^{(r-1)}; A_{(r-1)j})_{1,n_{r-1}}],$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha^{(1)}_{(r-1)ji_{r-1}},\cdots,\alpha^{(r-1)}_{(r-1)ji_{r-1}};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.5}$$

$$\mathbf{A}=[(\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj},0;A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r},0;A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A=[(\mathrm{c}^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}];\cdots;$$

$$[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_j)_{n^{(r)}+1,p^{(r)}_i}] \tag{1.7}$$

$$\mathbb{B}=[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta^{(1)}_{(r-1)ji_{r-1}},\cdots,\beta^{(r-1)}_{(r-1)ji_{r-1}};B_{(r-1)ji_{r-1}})_{1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B}=[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r},0;B_{rji_r})_{1,q_{i_r}}] \tag{1.9}$$

$$\mathrm{B}=[(\mathrm{d}^{(1)}_j,\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}];\cdots;$$

$$[(\mathrm{d}^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_j)_{m^{(r)}+1,q^{(r)}_i}] \tag{1.10}$$

$$U=0,n_2;0,n_3;\cdots;0,n_{r-1};V=m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X=p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

Srivastava ([6], p. 1, Eq. (1)) have introduced the general class of polynomials :

$$S^M_N(x)=\sum_{K=0}^{[N/M]}\frac{(-N)_{MK}}{K!}A_{N,K}x^K \tag{1.13}$$

where $M$ is an arbitrary positive integer and the coeficients $A_{N,K}$ are arbitrary constants real or complex. On specializing these coefficients $A_{N,K}$, $S^M_N[.]$ yields a number of known polynomials as special cases. These include, among others, Hermite polynomials, Jacobi polynomials, Laguerre polynomials, bessel polynomials and several others([10], p. 158-161).

We shall note

$$a_{NK}=\frac{(-N)_{MK}}{K!}A_{N,K} \tag{1.14}$$

## 2. Main integrals.

In this section, we evaluate two unified integrals.

**Theorem 1.**

$$\int_0^b x^{\rho-1}(b-x)^{\upsilon-1}(x^t+a^t)^{-\lambda}\daleth\left(\begin{array}{c}\mathrm{z}_1x^{u_1}(b-x)^{v_1}(x^t+a^t)^{-w_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_rx^{u_r}(b-x)^{v_r}(x^t+a^t)^{-w_r}\end{array}\right)\mathrm{d}x=a^{-\lambda t}b^{\rho+\upsilon-1}\daleth^{U;0,n_r+3:V;1,0}_{X;p_{i_r}+3,q_{i_r}+2,\tau_{i_r}:R_r:Y;0,1}$$

$$\left(\begin{array}{c|c} \mathrm{z}_1 a^{-tw_1} b^{u_1+v_1} & \mathbb{A};\ (1\text{-}\rho; u_1,\cdots,u_r,t;1),(1-\upsilon;v_1,\cdots,v_r,0;1),(1-\lambda;w_1,\cdots,w_r,1;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r a^{-tw_r} b^{u_r+v_r} & \\ \left(\frac{b}{a}\right)^t & \mathbb{B};\ \mathbf{B},(1\text{-}\rho-\upsilon;u_1+v_1,\cdots,u_r+v_r,t;1),(1-\lambda;w_1,\cdots,w_r,0;1):B;(0,1;1) \end{array}\right) \tag{2.1}$$

provided

$$Re(\lambda), u_i, v_i, w_i > 0 (i=1,\cdots,r),\ Re(\rho)+\sum_{i=1}^r u_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left(D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0.$$

$$Re(\upsilon)+\sum_{i=1}^r v_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left(D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0.$$

$|arg(x^{u_i}(b-x)^{v_i}(x^t+a^t)^{-w_i}z_i)| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4). $\left|\frac{b}{a}\right| < 1.$

Proof

To prove the integral (2.1), we first express the multivariable Gimel function occurring on its left-hand side in terms of Mellin-Barnes multiple integral contours and then interchange the order of $x$ and $(s_1,\cdots,s_r)$ integrals (which is permissible under the conditions mentioned in (2.1)), the left-hand side of (2.1) (say I) takes the following form :

$$I=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{i=1}^r\theta_i(s_i)z_i^{s_i}\int_0^b x^{\rho+\sum_{i=1}^r u_is_i-1}(b-x)^{\upsilon+\sum_{i=1}^r v_is_i-1}(x^t+a^t)^{-\lambda-\sum_{i=1}^r w_is_i}\mathrm{d}x$$

$$\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.2}$$

Now if we express the term $(x^t+a^t)^{-\lambda-\sum_{i=1}^r w_is_i}$ occurring in the $x$ integral of the equation given in terms of Mellin-Barnes multiple integrals contour ([7], p. 18, Eq. (2.6.4); p. 10, Eq. (2.1.1)) the equation (2.2) takes the following form

$$I=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{i=1}^r\theta_i(s_i)z_i^{s_i}\int_0^b x^{\rho+\sum_{i=1}^r u_is_i-1}(b-x)^{\upsilon+\sum_{i=1}^r v_is_i-1}$$

$$\left[\frac{a^{-(\lambda+\sum_{i=1}^r w_is_i)t}}{\Gamma(\lambda+\sum_{i=1}^r w_is_i)}\frac{1}{2\pi\omega}\int_{L_{r+1}}\Gamma\left(\sum_{i=1}^r w_is_i+s_{r+1}\right)\Gamma(-s_{r+1})\left(\frac{x}{a}\right)^{ts_{r+1}}\mathrm{d}s_{r+1}\right]\mathrm{d}x\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.3}$$

Interchanging the order of $x$ and $s_{r+1}$ integrals (which is permissible under the conditions mentioned in (2.1)), and evaluating the $x$ integral thus obtained we get after algebraic manipulations

$$I=\frac{b^{\rho+\upsilon-1}}{(2\pi\omega)^{r+1}a^{\lambda t}}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{i=1}^r\theta_i(s_i)z_i^{s_i}\left(a^{-tw_i}b^{u_i+v_i}\right)^{s_i}\left(\frac{b^t}{a^t}\right)^{s_{r+1}}$$

$$\frac{\Gamma(\rho+\sum_{i=1}^r u_is_i+ts_{r+1})\Gamma(\upsilon+\sum_{i=1}^r v_is_i)}{\Gamma(\rho+\upsilon+\sum_{i=1}^r(u_i+v_i)s_i+ts_{r+1})}\frac{\Gamma(\lambda+\sum_{i=1}^r w_is_i+s_{r+1})\Gamma(-s_{r+1})}{\Gamma(\lambda+\sum_{i=1}^r w_is_i)}\mathrm{d}s_1\cdots\mathrm{d}s_r\mathrm{d}s_{r+1} \tag{2.4}$$

On reinterpreting the multiple Mellin-barnes integrals contour occurring in (2.4) in term of the Gimel-function of $(r+1)$-variables, we get the desired integral (2.1).

**Theorem 2.**

$$\int_0^b x^{\rho-1}(b-x)^{\upsilon-1}(x^t+a^t)^{-\lambda}S_N^M\left(x^u(b-x)^v(x^t+a^t)^{-w}\right)S_{N'}^{M'}\left(x^{u'}(b-x)^{v'}(x^t+a^t)^{-w'}\right)$$

$$\beth\begin{pmatrix} z_1x^{u_1}(b-x)^{v_1}(x^t+a^t)^{-w_1} \\ . \\ . \\ . \\ z_rx^{u_r}(b-x)^{v_r}(x^t+a^t)^{-w_r} \end{pmatrix}\mathrm{d}x = a^{-\lambda t}b^{\rho+\upsilon-1}\sum_{K=0}^{[N/M]}\sum_{K'=0}^{[N'/M']}a^{-(wK+w'K')}b^{(u+v)K+(u'+v')K'}$$

$$d^Ke^{K'}a_{NK}a'_{N'K'}\beth^{U;0,n_r+3:V;1,0}_{X;p_{i_r}+3,q_{i_r}+2,\tau_{i_r}:R_r:Y;0,1}\left(\begin{matrix} z_1a^{-tw_1}b^{u_1+v_1} \\ . \\ . \\ z_ra^{-tw_r}b^{u_r+v_r} \\ \left(\frac{b}{a}\right)^t \end{matrix}\middle|\begin{matrix} \mathbb{A};\mathrm{A}_2,\mathbf{A}:A \\ . \\ . \\ \mathbb{B};\mathbf{B},\mathrm{B}_2:B;(0,1;1) \end{matrix}\right) \tag{2.5}$$

where

$$A_2=(1-\rho-uK-u'K';u_1,\cdots,u_r,t;1),(1-\upsilon-vK-v'K';v_1,\cdots,v_r,0;1),(1-\lambda-wK-w'K';w_1,\cdots,w_r,1;1) \tag{2.6}$$

$$B_2=(1-\rho-\upsilon-(u+v)K-(u'+v')K';u_1+v_1,\cdots,u_r+v_r,t;1),(1-\lambda-wK-w'K';w_1,\cdots,w_r,0;1) \tag{2.7}$$

provided

$Re(\lambda),u,v,w,u',v',w',u_i,v_i,w_i>0(i=1,\cdots,r)$, $Re(\rho)+\sum_{i=1}^{r}u_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left(D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0.$

$Re(\upsilon)+\sum_{i=1}^{r}v_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left(D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0.$

$|arg(x^{u_i}(b-x)^{v_i}(x^t+a^t)^{-w_i}z_i)|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4). $\left|\frac{b}{a}\right|<1.$

Proof
We first express both the general class of polynomials occurring in the integrand (2.5) in the series form with the help of (1.13) and then interchange the order of summations and integration (which is permissible under the conditions stated in (2.5)) so that the left-hand side of the integral (2.5) (say J) assume the following form :

$$J=\sum_{K=0}^{[N/M]}\sum_{K'=0}^{[N'/M']}d^Ke^{K'}\int_0^b x^{\rho+uK+u'K'-1}(b-x)^{\upsilon+vK+V'K'-1}(x^t+a^t)^{-\lambda-wK-w'K'}$$

$$\beth\begin{pmatrix} z_1x^{u_1}(b-x)^{v_1}(x^t+a^t)^{-w_1} \\ . \\ . \\ . \\ z_rx^{u_r}(b-x)^{v_r}(x^t+a^t)^{-w_r} \end{pmatrix}\mathrm{d}x \tag{2.8}$$

Evaluating the $x$-integral occurring in the above formula with the help of the theorem 1, wa get the desired result.

## 3. Special cases.

If we take $N'=0$, we get the following integral

**Corollary 1.**

$$\int_0^b x^{\rho-1}(b-x)^{\upsilon-1}(x^t+a^t)^{-\lambda}S_N^M\left(x^u(b-x)^v(x^t+a^t)^{-w}\right)\gimel\begin{pmatrix} z_1x^{u_1}(b-x)^{v_1}(x^t+a^t)^{-w_1}\\ \cdot\\ \cdot\\ \cdot\\ z_rx^{u_r}(b-x)^{v_r}(x^t+a^t)^{-w_r}\end{pmatrix}dx = a^{-\lambda t}b^{\rho+\upsilon-1}$$

$$\sum_{K=0}^{[N/M]} a^{-wK}b^{(u+v)K}d^K a_{NK}\,\gimel^{U;0,n_r+3:V;1,0}_{X;p_{i_r}+3,q_{i_r}+2,\tau_{i_r}:R_r:Y;0,1}\left(\begin{matrix} z_1a^{-tw_1}b^{u_1+v_1}\\ \cdot\\ \cdot\\ z_ra^{-tw_r}b^{u_r+v_r}\\ \left(\frac{b}{a}\right)^t\end{matrix}\,\middle|\,\begin{matrix}\mathbb{A};\ A_2',\mathbf{A}:A\\ \cdot\\ \cdot\\ \mathbb{B};\ \mathbf{B},B_2':B;(0,1;1)\end{matrix}\right) \quad (3.1)$$

where

$$A_2'=(1-\rho-uK;u_1,\cdots,u_r,t;1),(1-\upsilon-vK-;v_1,\cdots,v_r,0;1),(1-\lambda-wK;w_1,\cdots,w_r,1;1) \quad (3.2)$$

$$B_2'=(1-\rho-\upsilon-(u+v)K;u_1+v_1,\cdots,u_r+v_r,t;1),(1-\lambda-wK;w_1,\cdots,w_r,0;1) \quad (3.3)$$

under the same existence conditions that (2.5).

If we take $v=w=0=v_i=w_i(i=1,\cdots,r)$, and reinterpreting the new multivariable Gimel-function occurring on its right-hand side, we arrive at the following result :

**Corollary 2.**

$$\int_0^b x^{\rho-1}(b-x)^{\upsilon-1}(x^t+a^t)^{-\lambda}S_N^M\left(dx^u\right)\gimel\begin{pmatrix} z_1x^{u_1}\\ \cdot\\ \cdot\\ \cdot\\ z_rx^{u_r}\end{pmatrix}dx = a^{-\lambda t}b^{\rho+\upsilon-1}\frac{\Gamma(\upsilon)}{\Gamma(\lambda)}\sum_{K=0}^{[N/M]} d^Kb^{uK}d^Ka_{NK}$$

$$\gimel^{U;0,n_r+1:V;1,1}_{X;p_{i_r}+1,q_{i_r}+1,\tau_{i_r}:R_r:Y;1,1}\left(\begin{matrix} z_1b^{u_1}\\ \cdot\\ \cdot\\ z_rb^{u_r}\\ \left(\frac{b}{a}\right)^t\end{matrix}\,\middle|\,\begin{matrix}\mathbb{A};\ (1\text{-}\rho-uK;u_1,\cdots,u_r,t;1),\mathbf{A}:A;(\lambda,1;1)\\ \cdot\\ \cdot\\ \mathbb{B};\ \mathbf{B},(1\text{-}\rho-\upsilon-uK;u_1,\cdots,u_r,t;1):B;(0,1;1)\end{matrix}\right) \quad (3.2)$$

under the same existence conditions that (2.5).

If we take $d=u=1, M=1$ and $A_{N,K}=\dfrac{\Gamma(1+\alpha+\beta N)}{N!\Gamma(1+\alpha+\beta K)}$ in the above integral, the polynomial $s_N^1[.]$ reduces to $Z_N^\alpha\left(x^{\frac{1}{\beta}};\beta\right)$(Konhauser biorthogonal polynomials ([3], p. 304)), we obtain the following integral

**Corollary 3.**

$$\int_0^b x^{\rho-1}(b-x)^{\upsilon-1}(x^t+a^t)^{-\lambda}Z_N^\alpha\left(x^{\frac{1}{\beta}}\right)\gimel\begin{pmatrix} z_1x^{u_1}\\ \cdot\\ \cdot\\ \cdot\\ z_rx^{u_r}\end{pmatrix}dx = a^{-\lambda t}b^{\rho+\upsilon-1}\frac{\Gamma(\upsilon)}{\Gamma(\lambda)}\sum_{K=0}^{N}\frac{b^K(-N)_K\Gamma(1+\alpha+\beta N)}{K!N!\Gamma(1+\alpha+\beta K)}$$

$$\gimel^{U;0,n_r+1:V;1,1}_{X;p_{i_r}+1,q_{i_r}+1,\tau_{i_r}:R_r:Y;1,1}\left(\begin{matrix} z_1b^{u_1}\\ \cdot\\ \cdot\\ z_rb^{u_r}\\ \left(\frac{b}{a}\right)^t\end{matrix}\,\middle|\,\begin{matrix}\mathbb{A};\ (1\text{-}\rho-uK;u_1,\cdots,u_r,t;1),\mathbf{A}:A;(\lambda,1;1)\\ \cdot\\ \cdot\\ \mathbb{B};\ \mathbf{B},(1\text{-}\rho-\upsilon-uK;u_1,\cdots,u_r,t;1):B;(0,1;1)\end{matrix}\right) \quad (3.3)$$

under the same existence conditions that (2.5).

Taking $\beta = 1$, the konhauser polynomial reduces in Laguerre polynomials $L_N^{(\alpha)}(x)$ and we obtain

**Corollary 4.**

$$\int_0^b x^{\rho-1}(b-x)^{\upsilon-1}(x^t+a^t)^{-\lambda}L_N^{\alpha}(x)\gimel\begin{pmatrix} z_1x^{u_1}\\ \cdot\\ \cdot\\ \cdot\\ z_rx^{u_r}\end{pmatrix}\mathrm{d}x = a^{-\lambda t}b^{\rho+\upsilon-1}\frac{\Gamma(\upsilon)}{\Gamma(\lambda)}\sum_{K=0}^{N}\frac{b^K(-N)_K\Gamma(1+\alpha+N)}{K!N!\Gamma(1+\alpha+K)}$$

$$\gimel^{U;0,n_r+1:V;1,1}_{X;p_{i_r}+1,q_{i_r}+1,\tau_{i_r}:R_r:Y;1,1}\left(\begin{array}{c|c} z_1b^{u_1} & \mathbb{A};\ (1\text{-}\rho-uK;u_1,\cdots,u_r,t;1),\mathbf{A}:A;(\lambda,1;1)\\ \cdot & \cdot\\ \cdot & \cdot\\ z_rb^{u_r} & \mathbb{B};\ \mathbf{B},(1\text{-}\rho-\upsilon-uK;u_1,\cdots,u_r,t;1):B;(0,1;1)\\ \left(\frac{b}{a}\right)^t & \end{array}\right) \tag{3.4}$$

under the same existence conditions that (2.5).

If we take $v_i = w_i = 0 = v = w(i = 1,\cdots,r), d = u = 1 = m$, $A_{N,K} = \binom{N+\alpha}{N}\frac{1}{(\alpha+1)_K}$ and $e = u' = 1, m' = 2$, $v' = w' = 0, A_{N',K'} = (-)^{K'}$, $S_N^M(x)$ occurring therein breaks into the Laguerre polynomials ([10], p.159, Eq. (1.8)) and $S_{N'}^{M'}(x)$ reduces to the Hermite polynomials ([10], p.158, Eq. (1.4)) and the integral (2.5) takes the following form :

**Corollary 5.**

$$\int_0^b x^{\rho-1}(b-x)^{\upsilon-1}(x^t+a^t)^{-\lambda}L_N^{\alpha}(x)H_{N'}\left(\frac{1}{2\sqrt{x}}\right)\gimel\begin{pmatrix} z_1x^{u_1}\\ \cdot\\ \cdot\\ \cdot\\ z_rx^{u_r}\end{pmatrix}\mathrm{d}x = a^{-\lambda t}b^{\rho+\upsilon-1}\frac{\Gamma(\upsilon)}{\Gamma(\lambda)}\sum_{K=0}^{N}\sum_{K'=0}^{[N'/2]}\frac{b^{K+K'}(-N)_K(-N')_{2K'}}{K!K'!}$$

$$\binom{N+\alpha}{N}\frac{(-)^{K'}}{(\alpha+1)_K}\gimel^{U;0,n_r+1:V;1,1}_{X;p_{i_r}+1,q_{i_r}+1,\tau_{i_r}:R_r:Y;1,1}\left(\begin{array}{c|c} z_1b^{u_1} & \mathbb{A};\ (1\text{-}\rho-uK;u_1,\cdots,u_r,t;1),\mathbf{A}:A;(\lambda,1;1)\\ \cdot & \cdot\\ \cdot & \cdot\\ z_rb^{u_r} & \mathbb{B};\ \mathbf{B},(1\text{-}\rho-\upsilon-uK;u_1,\cdots,u_r,t;1):B;(0,1;1)\\ \left(\frac{b}{a}\right)^t & \end{array}\right) \tag{3.5}$$

under the same existence conditions that (2.5).

## 4. Conclusion.

The integrals (2.1) and (2.5) established here are unified and act as key formulae. Thus the multivariable Gimel-function occurring in these integrals can be suitably specialized to a remarkably wide variety of special functions (or product of several such special functions) which are expressible in terms of E, F, G, H, I, Aleph-functions of one or more variables. Again the class of polynomials involved in the integral (2.5) reduces to a large number of polynomials listed by Srivastava and Singh ([10], p.158-161), therefore, from the integral (2.5) we can further obtain various integrals involving a number of simpler polynomials.

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