A Study on Comparison of Jacobi, Gauss-Seidel and Sor Methods for the Solution in System of Linear Equations

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Abstract — *This paper presents three iterative methods for the solution of system of linear equations has been evaluated in this work. The result shows that the Successive Over-Relaxation method is more efficient than the other two iterative methods, number of iterations required to converge to an exact solution. This research will enable analyst to appreciate the use of iterative techniques for understanding the system of linear equations.*

Keywords — *The system of linear equations, Iterative methods, Initial approximation, Jacobi method, Gauss-Seidel method, Successive Over- Relaxation method.*

1. INTRODUCTION AND PRELIMINARIES

Numerical analysis is the area of mathematics and computer science that creates, analyses, and implements algorithms for solving numerically the problems of continuous mathematics. Such problems originate generally from real-world applications of algebra, geometry and calculus, and they involve variables which vary continuously. These problems occur throughout the natural sciences, social science, engineering, medicine, and business.

The solution of system of linear equations can be accomplished by a numerical method which falls in one of two categories: direct or iterative methods. We have so far discussed some direct methods for the solution of system of linear equations and we have seen that these methods yield the solution after an amount of computation that is known advance.

We shall now describe the iterative or indirect methods, which start from an approximation to the true solution and, if convergent, derive the sequence of closer approximations-the cycle of computations being repeated till the required accuracy is obtained. This means that in a direct method the amount of computation is fixed, while in an iterative method the amount of computation depends on the accuracy required.

In general, one should prefer a direct method for the solution a linear system, but in the case of matrices with a large number of zero elements, it will be advantageous to use iterative methods which presents these elements.

2. PROBLEM FORMULATION

In this section, we consider the system of 'n' linear equations in 'n' unknowns is given by

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{array}\right\} \quad \ldots\ldots\ldots\ (1)$$

May be represented as the matrix equation, where

$$AX=B \quad \ldots\ldots\ldots\ldots\ldots\ (2)$$

Where

$$A = \begin{bmatrix} a11 & \cdots & a1n \\ \vdots & \ddots & \vdots \\ an1 & \cdots & ann \end{bmatrix} \quad X = \begin{pmatrix} x1 \\ \vdots \\ xn \end{pmatrix} \quad \text{And} \quad B = \begin{pmatrix} b1 \\ \vdots \\ bn \end{pmatrix}$$

In which the diagonal elements a_{ij} do not vanish. If this is not the case, then the equations should be rearranged, so that this condition is satisfied. Now we rewrite the system of equation (1) as

$$x_1 = \frac{b_1 - (a_{12}x_2 + \cdots + a_{1n}x_n)}{a_{11}}$$
$$x_2 = \frac{b_2 - (a_{21}x_1 + \cdots + a_{2n}x_n)}{a_{22}}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots....$$
$$x_n = \frac{b_n - (a_{n1}x_1 + a_{n2}x_2 \ldots + a_{nn-1}x_{n-1})}{a_{nn}}$$

In general, we have

$$x_i = \frac{b_i - (\sum_{j \neq i} a_{ij}x_j)}{a_{ii}} \quad \ldots\ldots\ldots\ldots (3)$$

Now, if an initial guess $x^0_1, x^0_2, x^0_3 \ldots\ldots x^0_n$, for all the unknowns was available, we could substitute these values into the right-hand side of the set of equations (3) and compute an updated guess for the unknowns, $x^1_1, x^1_2, x^1_3 \ldots\ldots x^1_n$. There are several ways to accomplish this, depending on how you use the most recently computed guesses.

3. ITERATIVE METHODS

The approximate methods for solving system of linear equations makes it possible to obtain the values of the roots system with the specified accuracy. This process of constructing such a sequence is known as iteration. Three closely related method studied in this work are all iterative in nature. Unlike the direct methods, which attempts to calculate an exact solution in a finite number of operations, these methods starts with an initial approximation and generate successively improved approximations in an infinite sequence whose limit is the exact solution. In practical terms, this has more advantage, because the direct solution will be subject to rounding errors.

3.1 JACOBI METHOD

In numerical linear algebra, the Jacobi method is an algorithm for determining the solutions of a diagonally dominant system of linear equations. Each diagonal element is solved for, and an approximate value is plugged in. The process is then iterated until it converges. This algorithm is a stripped- down version of the Jacobi transformation method of matrix diagonalization.

In the Jacobi method, all of the values of the unknowns are updated before any of the new information is used in the calculations. That is, starting with the initial guess $x^0_1, x^0_2, x^0_3 \ldots\ldots x^0_n$, compute the next approximation of the solution as

$$x_1^1 = \frac{b_1 - (a_{12}x_2^0 + \cdots + a_{1n}x_n^0)}{a_{11}}$$
$$x_2^1 = \frac{b_2 - (a_{21}x_1^0 + \cdots + a_{2n}x_n^0)}{a_{22}}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$x_n^1 = \frac{b_n - (a_{n1}x_1^0 + a_{n2}x_2^0 + \cdots + a_{nn-1}x_{n-1}^0)}{a_{nn}}$$

Or, after k iterations of this process, we have

$$x_1^{k+1} = \frac{b_1 - (a_{12}x_2^k + \cdots + a_{1n}x_n^k)}{a_{11}}$$
$$x_2^{k+1} = \frac{b_2 - (a_{21}x_1^k + \cdots + a_{2n}x_n^k)}{a_{22}}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$x_n^{k+1} = \frac{b_n - (a_{n1}x_1^k + a_{n2}x_2^k + \cdots + a_{nn-1}x_{n-1}^k)}{a_{nn}}$$

More generally

$$x_i^{k+1} = \frac{b_i - (\sum_{j \neq i} a_{ji}x_j^k)}{a_{ii}} \quad \ldots\ldots\ldots\ldots (4)$$

This method is due to Jacobi and is called the method of simultaneous displacements.

3.2 GAUSS SEIDEL METHOD

In the first equation of equation (1), we substitute the first iterations $x^0_1, x^0_2, x^0_3 \ldots\ldots x^0_n$, into the right-hand side and denote the results as x^1_1. In the second equation we substitute $x^1_1, x^0_2, x^0_3 \ldots\ldots x^0_n$, and denote the result as x^1_2. In this manner, we complete the first stage of iteration and the entire processes is repeated till the values of $x_1, x_2, x_3 \ldots\ldots x_n$ are obtained to the accuracy required. It is clear, therefore, that this method used an improved component as shown as it is available and it is called the method of successive displacement (or) the Gauss-seidel method. If we implement this, our method would look like

$$x_1^1 = \frac{b_1 - (a_{12}x_2^0 + \cdots + a_{1n}x_n^0)}{a_{11}}$$

$$x_2^1 = \frac{b_2 - (a_{21}x_1^1 + \cdots + a_{2n}x_n^0)}{a_{22}}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_n^1 = \frac{b_n - (a_{n1}x_1^1 + a_{n2}x_2^1 + \cdots + a_{nn-1}x_{n-1}^1)}{a_{nn}}$$

Or, after k iterations of this process, we have

$$x_1^{k+1} = \frac{b_1 - (a_{12}x_2^k + \cdots + a_{1n}x_n^k)}{a_{11}}$$

$$x_2^{k+1} = \frac{b_2 - (a_{21}x_1^{k+1} + \cdots + a_{2n}x_n^k)}{a_{22}}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_n^{k+1} = \frac{b_n - (a_{n1}x_1^{k+1} + a_{n2}x_2^{k+1} + \cdots + a_{nn-1}x_{n-1}^{k+1})}{a_{nn}}$$

More generally

$$x_i^{k+1} = \frac{b_i - \left(\sum_{\substack{j<i \\ j\neq i}} a_{ji}x_j^{k+1} + \sum_{\substack{j>i \\ j\neq i}} a_{ji}x_j^k\right)}{a_{ii}} \quad \ldots\ldots\ldots\ldots (5)$$

3.3 SUCCESSIVE OVER-RELAXATION (SOR)

In numerical linear algebra, the method of successive over-relaxation (SOR) is a variant of the Gauss-Seidel method for solving a system of linear equations, resulting in faster convergence. A similar method can be used for any slowly converging iterative process.

The successive over-relaxation (SOR), is devised by applying extrapolation to the Gauss-Seidel method. This extrapolation takes the form of a weighted average between the previous iterate and the computed iterate successively for each component.

$$x_i^{(k+1)} = (1-\omega)\, x_i^k + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j<i} a_{ij}x_j^{k+1} - \sum_{j>i} a_{ij}x_j^k\right),\ i=1,2,3\ldots..n \ \ldots\ldots\ldots (6)$$

(where x denote a Gauss- Seidel iterate, and ω is the extrapolation factor). The idea is to choose a value for ω that will accelerate the rate of convergence of iterates to the solution.

If ω =1, the SOR method simplifies to the Gauss-Seidel method. Though technically the term under relaxation should be used when 0 < ω <1, for convenience the term over relaxation is now used for any value of ω Є (0,2).

4. CONVERGENCE OF ITERATIVE METHODS

The iterative methods converge, for any choice of the first approximation x^0_j (j=1,2,….), if every equation of the system (1) satisfies the condition that the sum of the absolute values of the coefficients a_{ij} / a_{ii} almost equal to, or in at least one equation less than unity.

$$\text{(i.e)}\quad |a_{ii}| \geq \sum |a_{ij}| \qquad \text{(i=1,2,3…..n)} \ \ldots\ldots\ldots (7)$$

Where the '<' sign should be valid in the case of 'at least' one equation. It can be shown that the Gauss-Seidel method converges twice as fast as the Jacobi method.

5. ANALYSIS OF RESULTS

The efficiency of the three iterative methods was compared based on a 2x2, 3x3 and a 4x4 order of linear equations. They are as follows from the examples

EXAMPLE -1 Solve the system

$$5x + y = 10$$
$$2x + 3y = 4$$

Using Jacobi, Gauss-Seidel and Successive Over-Relaxation methods.

SOLUTION

Given
$$5x + y = 10$$
$$2x + 3y = 4$$

The above equations can be written as the matrix form

$$AX = B$$

$$\begin{pmatrix} 5 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 4 \end{pmatrix}$$

Let

$$A = \begin{pmatrix} 5 & 1 \\ 2 & 3 \end{pmatrix}$$

the given matrix A is diagonally dominant (i.e $|\boldsymbol{a_{ii}}| \geq \sum|\boldsymbol{a_{ij}}|$),(ie $5\geq1$, $3\geq2$) hence, we apply the above said these iterative methods

To solve these equations by iterative methods, we are rewrite them as follows,

$$x=\frac{1}{5}(10 - y)$$
$$y=\frac{1}{3}(4 - 2x)$$

The results are given in the table -1(a)

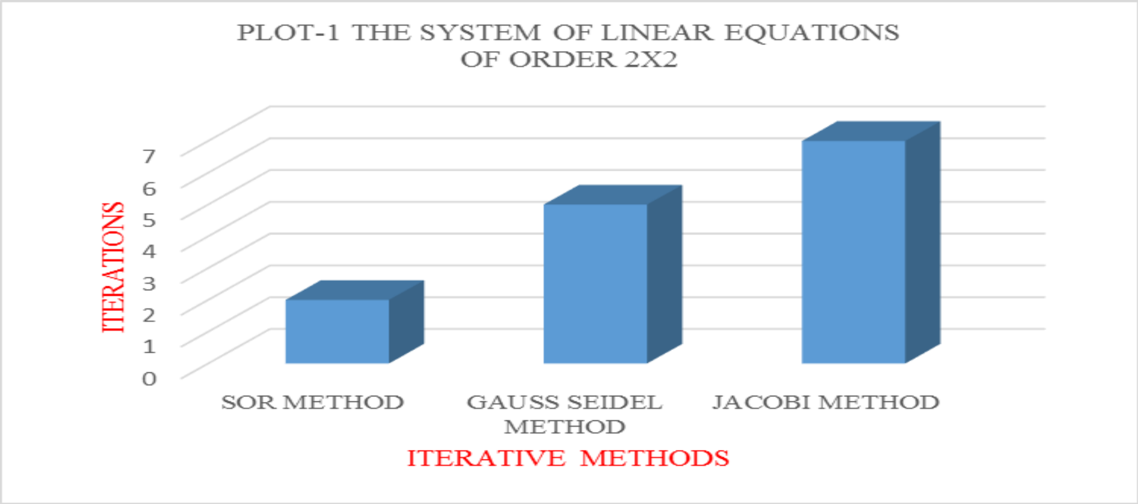
Table -1(a) Number of iterations of the iterative methods

JACOBI METHOD			GAUSS SEIDEL METHOD		SOR METHOD	
Iterations	x	Y	X	Y	x	Y
0	0	0	0	0	0	0
1	2	1.333	2	0	2.5	-0.4166
2	1.7333	0	2	0	2	0
3	2	0.1778	2	0		
4	1.9644	0	2	0		
5	2	0				
6	2	0				
7	2	0				

Table -1(b) Number of iterations for the SOR, GAUSS-SEIDEL AND JACOBI ITERATIVE METHODS

METHODS	NUBMER OF TERATIONS
SOR METHOD	2
GAUSS SEIDEL METHOD	5
JACOBI METHOD	7

Number of iterations for the SOR, GAUSS-SEIDEL AND JACOBI ITERATIVE METHODS **shown in the plot-1**



EXAMPLE – 2 Solve the system

$$10x + 2y - z = 7$$

$$1x + 8y + 3z = -4$$

$$-2x - y + 10z = 9$$

Using Jacobi, Gauss-Seidel and Successive Over-Relaxation methods.

SOLUTION

Given

$$10x + 2y - z = 7$$

$$1x + 8y + 3z = -4$$

$$-2x - y + 10z = 9$$

The above equations can be written as the matrix form

$$AX = B$$

Let

$$\begin{pmatrix} 10 & 2 & -1 \\ 1 & 8 & 3 \\ -2 & -1 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \\ 9 \end{pmatrix}$$

$$A = \begin{pmatrix} 10 & 2 & -1 \\ 1 & 8 & 3 \\ -2 & -1 & 10 \end{pmatrix}$$

the given matrix A is diagonally dominant (i.e $|a_{ii}| \geq \sum |a_{ij}|$), (ie10≥3, 8≥4, 10≥3) hence, we apply the above said these iterative methods

To solve these equations by iterative methods, we are rewrite them as follows,

$$x = \frac{1}{10}(7 - 2y + z)$$

$$y = \frac{1}{8}(-4 - x - 3z)$$

$$z = \frac{1}{10}(9 + 2x + y)$$

The results are given in the table 2(a)

Table 2(a) – Number of iterations of the iterative methods

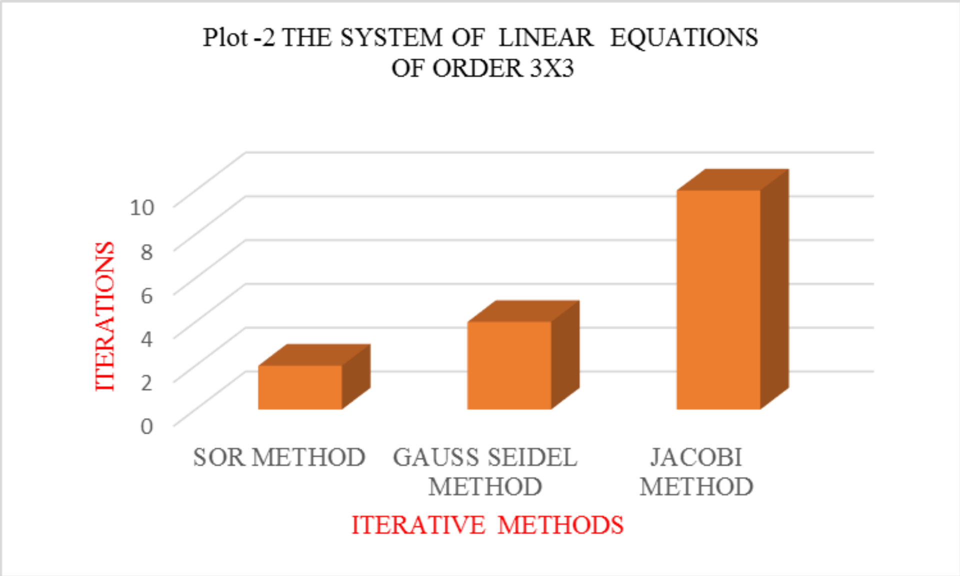
JACOBI METHOD				GAUSS-SEIDEL METHOD			SOR METHOD		
Iterations	X	Y	Z	x	Y	Z	X	Y	Z
0	0	0	0	0	0	0	0	0	0
1	0.7	-0.5	0.9	0.7	-0.5875	0.98125	0.77	-0.55	0.99
2	0.89	-0.925	0.99	0.9156	-0.9824	0.98488	0.9229	-1.0303	0.9807
3	0.984	-0.9825	0.9855	0.995	-0.9937	0.9996			
4	0.9951	-0.9926	0.9986	0.9987	-0.9997	1.1997			
5	0.9984	-0.9989	0.9998						
6	0.99976	-0.9997	0.9998						
7	0.9999	-0.9999	0.9999						
8	0.9999	-0.9999	0.9999						
9	0.9999	-0.9999	0.9999						
10	0.9999	-0.9999	0.9999						

Table -2(b) Number of iterations for the SOR, GAUSS-SEIDEL AND JACOBI ITERATIVE METHODS

METHODS	NUBMER OF ITERATIONS
SOR METHOD	2
GAUSS SEIDEL METHOD	4
JACOBI METHOD	10

Number of iterations for the SOR, GAUSS-SEIDEL AND JACOBI ITERATIVE METHODS

Shown in the plot-2



EXAMPLE – 3 Solve the system

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$

$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$

$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$

$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$

Using Jacobi, Gauss-Seidel and Successive Over-Relaxation methods.

SOLUTION

Given $10x_1 - 2x_2 - x_3 - x_4 = 3$

$-2x_1 + 10x_2 - x_3 - x_4 = 15$

$-x_1 - x_2 + 10x_3 - 2x_4 = 27$

$-x_1 - x_2 - 2x_3 + 10x_4 = -9$

The above equations can be written as the matrix form

AX = B

Let

$$\begin{pmatrix} 10 & -2 & -1 & -1 \\ -2 & 10 & -1 & -1 \\ -1 & -1 & 10 & -2 \\ -1 & -1 & -2 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 15 \\ 27 \\ -9 \end{pmatrix}$$

$$A = \begin{pmatrix} 10 & -2 & -1 & -1 \\ -2 & 10 & -1 & -1 \\ -1 & -1 & 10 & -2 \\ -1 & -1 & -2 & 10 \end{pmatrix}$$

the given matrix A is diagonally dominant (i.e $|a_{ii}| \geq \sum |a_{ij}|$),(ie10≥4, 10≥4, 10≥4, 10≥4) hence, we apply the above said these iterative methods

To solve these equations by iterative methods, we are rewrite them as follows,

$$x_1 = \frac{1}{10}(3+2x_2+x_3+x_4)$$

$$x_2 = \frac{1}{10}(15+2x_1+x_3+x_4)$$

$$x_3 = \frac{1}{10}(27+x_1+x_2+2x_4)$$

$$x_4 = \frac{1}{10}(-9+x_1+x_2+2x_3)$$

The results are given in the table 3(a)

Table 3(a)- Number of iterations of iterative methods

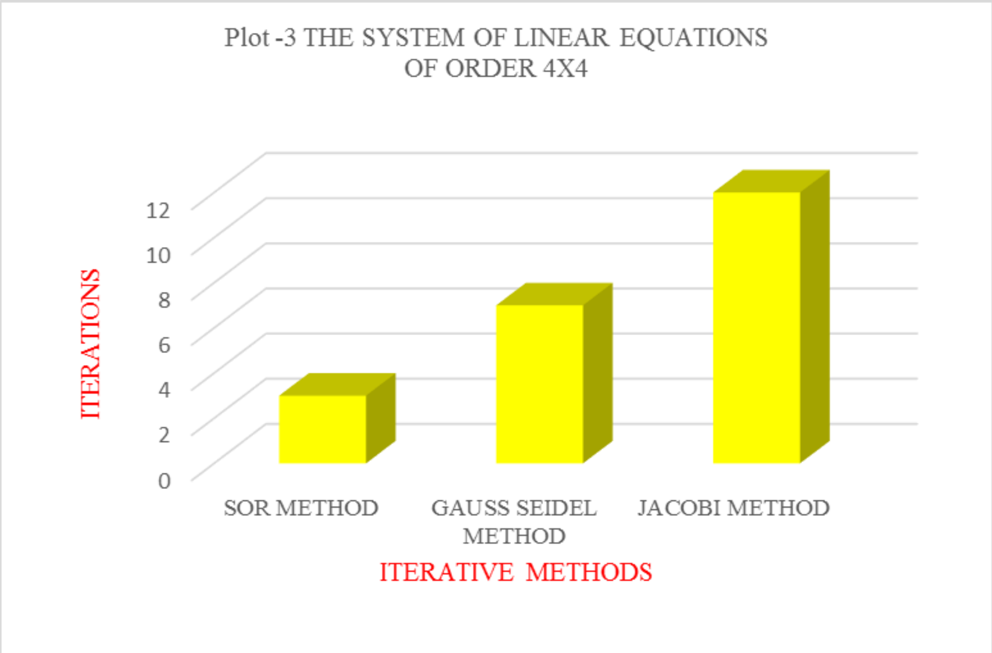
	JACOBI METHOD				**GAUSS - SEIDEL METHOD**				**SOR METHOD**			
Iterations	x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0.3	1.5	2.7	-0.9	0.3	1.56	2.886	-0.1368	0.375	1.96875	3.6679	0.0849
2	0.78	1.74	2.7	-0.18	0.8869	1.9523	2.9566	-0.0248	1.2425	2.1625	3.0	0.0
3	0.9	1.9608	2.916	-0.108	0.9836	1.9899	2.9924	-0.0042				
4	0.9624	1.9608	2.9592	-0.036	0.9968	1.9982	2.9987	-0.0008				
5	0.9845	1.9848	2.9851	-0.0158	0.9994	1.9997	2.9998	-0.0001				
6	0.9939	1.9938	2.9938	-0.006	0.9999	1.9999	3.0	0.0				
7	0.9975	1.9975	2.9976	-0.0025	1.0	2.0	3.0	0.0				
8	0.9990	1.9990	2.9990	-.0010								
9	0.9996	1.9996	2.9996	-0.0004								
10	0.9998	1.9998	2.9998	-0.0002								
11	0.9999	1.9999	2.9999	-0.0001								
12	1.0	2.0	3.0	0.0								

Table -3(b) Number of iterations for the SOR, GAUSS-SEIDEL AND JACOBI ITERATIVE METHODS

METHODS	NUMBER OF ITERATIONS
SOR METHOD	3
GAUSS SEIDEL METHOD	7
JACOBI METHOD	12

Number of iterations for the SOR, GAUSS-SEIDEL AND JACOBI ITERATIVE METHODS

Shown in the plot-3

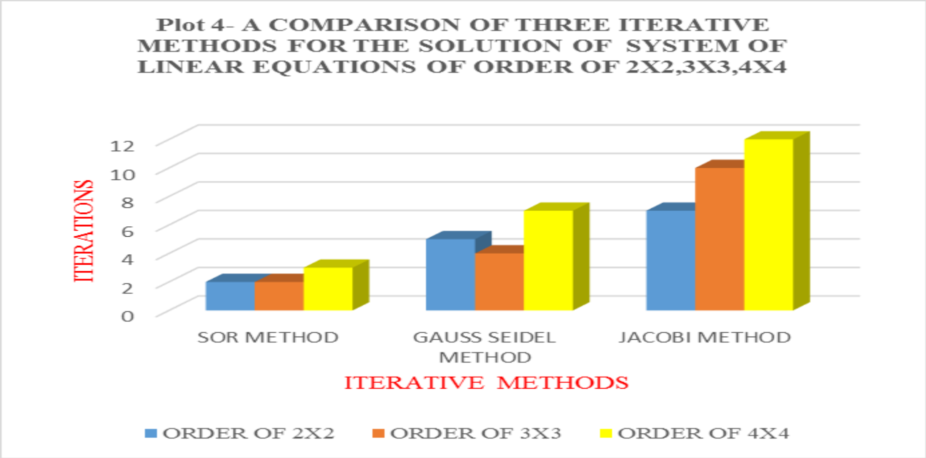


COMPARISION AND RESULTS

The three main iterative methods for solving linear equation have been presented, these are Successive Over Relaxation, The Gauss-Seidel and the Jacobi iterative. Three practical examples were analyzed, the order of 2x2, order of 3x3, order of 4x4 system of linear equations, even though the iterations can accommodate up to 10x10 system of linear equations. The analysis of results shows that Jacobi method takes longer iterations of 7,10,12 and for the order 2x2, order 3x3 and order 4x4 linear equations respectively. The number of iterations differ, as that of the Successive – Over Relaxation method of order 2x2, has 2 iterations, while Gauss-Seidel has 5 iterations. The number of iterations differ, as that of the Successive – Over Relaxation method of order 3x3, has 2 iterations, while Gauss-Seidel has 4 iterations. It also compare with the plot - 4 also.

Number of iterations for the SOR, GAUSS-SEIDEL AND JACOBI ITERATIVE METHODS

Shown in the plot-4



CONCLUSION

The number of iterations differ, as that of the Successive-Over Relaxation method of order 4x4 has 3 iterations, while Gauss-Seidel has 7 iterations. This shows that Successive-Over Relaxation requires less iteration than the Gauss-Seidel method. Thus, the Successive-Over Relaxation could be considered more efficient of the three methods.

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