An expansion formula for multivariable Gimel-function involving generalized Legendre Associated function

Frédéric Ayant

Teacher in High School , France

ABSTRACT

In this paper, we have evaluated an integral involving generalized Legendre's associated function and the multivariable Gimel-function defined here. This integral has been used in deriving an expansion formula for the multivariable Gimel-function in terms of a series of product of multivariable Gimel-function and generalized Legendre's associated function due to Meulenbeld [4].

KEYWORDS : Multivariable Gimel-function, multiple integral contours, expansion serie, Generalized Legendre's associated function.

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1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C}, \mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

We define a generalized transcendental function of several complex variables noted $\gimel$.

$$\gimel(z_1, \cdots, z_r) = \gimel^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}} \left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array} \right|$$

$$[(a_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}};[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},$$
$$[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{1,q_{i_2}};$$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(a_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{1,q_{i_3}};\cdots;\quad \cdots$$

$$[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{n_r+1,p_r}]:\ [(c^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]$$
$$[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{1,q_r}]:\ [(d^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]$$

$$\left. \begin{array}{l} ;\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_j)_{n^{(r)}+1,p^{(r)}_i}] \\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_j)_{m^{(r)}+1,q^{(r)}_i}] \end{array} \right)$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1, \cdots, s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k} \, ds_1 \cdots ds_r \tag{1.1}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha_{2ji_2}^{(k)}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta_{2ji2}^{(k)}s_k)]}$$

$$\frac{\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)}]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $n_2,\cdots,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2,0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant m_r,0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m_k);(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n_k);(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m_k);(k=1,\cdots,r)$.

$a_{kji_k}\in\mathbb{C};(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r)$.

$b_{kji_k}\in\mathbb{C};(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r)$.

$d_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = m^{(k)} + 1, \cdots, q_{i^{(k)}}); (k = 1, \cdots, r).$

$\gamma_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r).$

The contour L_k is in the $s_k(k = 1, \cdots, r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1 - a_{2j} + \sum_{k=1}^{2} \alpha_{2j}^{(k)} s_k\right) (j = 1, \cdots, n_2), \Gamma^{A_{3j}}\left(1 - a_{3j} + \sum_{k=1}^{3} \alpha_{3j}^{(k)} s_k\right)$ $(j = 1, \cdots, n_3), \cdots, \Gamma^{A_{rj}}\left(1 - a_{rj} + \sum_{i=1}^{r} \alpha_{rj}^{(i)}\right) (j = 1, \cdots, n_r), \Gamma^{C_j^{(k)}}\left(1 - c_j^{(k)} + \gamma_j^{(k)} s_k\right) (j = 1, \cdots, n^{(k)})(k = 1, \cdots, r)$to the right of the contour L_k and the poles of $\Gamma^{D_j^{(k)}}\left(d_j^{(k)} - \delta_j^{(k)} s_k\right) (j = 1, \cdots, m^{(k)})(k = 1, \cdots, r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2} A_i^{(k)} \pi$ where

$$A_i^{(k)} = \sum_{j=1}^{m^{(k)}} D_j^{(k)} \delta_j^{(k)} + \sum_{j=1}^{n^{(k)}} C_j^{(k)} \gamma_j^{(k)} - \tau_{i^{(k)}} \left(\sum_{j=m^{(k)}+1}^{q_i^{(k)}} D_{ji^{(k)}}^{(k)} \delta_{ji^{(k)}}^{(k)} + \sum_{j=n^{(k)}+1}^{p_i^{(k)}} C_{ji^{(k)}}^{(k)} \gamma_{ji^{(k)}}^{(k)} \right) +$$

$$-\tau_{i_2} \left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2} \alpha_{2ji_2}^{(k)} + \sum_{j=1}^{q_{i_2}} B_{2ji_2} \beta_{2ji_2}^{(k)} \right) - \cdots - \tau_{i_r} \left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r} \alpha_{rji_r}^{(k)} + \sum_{j=1}^{q_{i_r}} B_{rji_r} \beta_{rji_r}^{(k)} \right) \quad (1.4)$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \to 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \to \infty$ where $i = 1, \cdots, r$:

$\alpha_i = \min_{1 \leqslant j \leqslant m^{(i)}} Re \left[D_j^{(i)} \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right]$ and $\beta_i = \max_{1 \leqslant j \leqslant n^{(i)}} Re \left[C_j^{(i)} \left(\frac{c_j^{(i)} - 1}{\gamma_j^{(i)}} \right) \right]$

Remark 1.
If $n_2 = \cdots = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$ $A_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the multivariable Gimel-function reduces in the multivariable Aleph- function defined by Ayant [1].

Remark 2.
If $n_2 = \cdots = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1,$ then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [7].

Remark 3.
If$A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [6].

Remark 4.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H-function defined by Srivastava and Panda [8,9].

In your investigation, we shall use the following notations.

$\mathbb{A} = [(\mathrm{a}_{2j}; \alpha_{2j}^{(1)}, \alpha_{2j}^{(2)}; A_{2j})]_{1,n_2}, [\tau_{i_2}(a_{2ji_2}; \alpha_{2ji_2}^{(1)}, \alpha_{2ji_2}^{(2)}; A_{2ji_2})]_{n_2+1,p_{i_2}}, [(a_{3j}; \alpha_{3j}^{(1)}, \alpha_{3j}^{(2)}, \alpha_{3j}^{(3)}; A_{3j})]_{1,n_3},$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha^{(1)}_{(r-1)j},\cdots,\alpha^{(r-1)}_{(r-1)j};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha^{(1)}_{(r-1)ji_{r-1}},\cdots,\alpha^{(r-1)}_{(r-1)ji_{r-1}};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.5}$$

$$\mathbf{A}=[(\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A=[(\mathrm{c}^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}];\cdots;$$

$$[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_j)_{n^{(r)}+1,p^{(r)}_i}] \tag{1.7}$$

$$\mathbb{B}=[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta^{(1)}_{(r-1)ji_{r-1}},\cdots,\beta^{(r-1)}_{(r-1)ji_{r-1}};B_{(r-1)ji_{r-1}})_{1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B}=[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{1,q_{i_r}}] \tag{1.9}$$

$$\mathrm{B}=[(\mathrm{d}^{(1)}_j,\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}];\cdots;$$

$$[(\mathrm{d}^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,m_r}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_j)_{m_r+1,q^{(r)}_i}] \tag{1.10}$$

$$U=0,n_2;0,n_3;\cdots;0,n_{r-1};V=m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X=p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

2. Required results.

The following integral is required here given by Meulenbeld and Robin ([5], p. 343, Eq. (38))

Lemma 1.

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P^{m,n}_{k-\frac{m-n}{2}}(x)\mathrm{d}x=\frac{2^{\rho+\sigma-\frac{m-n}{2}+1}\Gamma\left(\rho-\frac{m}{2}+1\right)\Gamma\left(\sigma-\frac{n}{2}+1\right)}{\Gamma(1-m)\Gamma\left(\rho+\sigma-\frac{m-n}{2}+2\right)} \tag{2.1}$$

provided that : $Re\left(\rho-\frac{m}{2}\right)>-1,Re\left(\sigma+\frac{n}{2}\right)>-1.$

Orthogonality property of the generalized Legendre's function ([5], p. 340, Eq. (26) and (27))

Lemma 2.

$$\int_{-1}^{1}P^{u,v}_{k-\frac{u-v}{2}}(x)P^{u,v}_{l-\frac{u-v}{2}}(x)\mathrm{d}x=\left[\begin{array}{c}0 \text{ if k=1}\\ .\\ \frac{2^{v-u+1}k!\Gamma(k+v+1)}{(2k-u+v+1)\Gamma(k-u+1)\Gamma(k-u+v+1)} \text{ if k = l}\end{array}\right. \tag{2.2}$$

provided $Re(u)<1,Re(v)>-1.$

3. Main integral.

The integral to evaluated is

Theorem 1.

$$\int_{-1}^{1}(1-x)^{\rho-\frac{u}{2}}(1+x)^{\sigma+\frac{v}{2}}P^{m,n}_{k-\frac{m-n}{2}}(x)\gimel\left((1-x)^{h_1}(1+x)^{k_1}z_1,\cdots,(1-x)^{h_r}(1+x)^{k_r}z_r\right)\mathrm{d}x=$$

$$2^{\rho+\sigma-u+v+1}\sum_{t=0}^{\infty}\frac{(-l)_t(v-u+l+1)_t}{\Gamma(1-u+t)t!}$$

$$\gimel^{U;0,n_r+2:V}_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} 2^{h_1+k_1}z_1 & \mathbb{A};\ (\text{u-}\rho-t;h_1,\cdots,h_r;1),(-\sigma-v;k_1,\cdots,k_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{h_r+k_r}z_r & \mathbb{B};\ \mathbf{B},(\text{u-v-}\rho-\sigma-t-1;h_1+k_1,\cdots,h_r+k_r;1):B \end{array}\right) \tag{3.1}$$

provided

$h_i,k_i>0(i=1,\cdots,r)$, $Re(\rho-u)+\sum_{i=1}^{r}h_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left(D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-1,$

$Re(\sigma+v)+\sum_{i=1}^{r}k_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left(D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-1$. $|arg(z_i(1-x)^{h_i}(1+x)^{k_i})|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Proof

Substituting the expression of the multivariable Gimel-function in terms of Mellin-Barnes multiple integrals contour with the help of (1.1) and interchanging the order of integrations, (which is permissible under the conditions mentioned in (3.1)). On evaluating the inner integral with the help of the lemma 1 and interpreting the Mellin-Barnes multiple integrals contour in terms of the multivariable Gimel-function, we get the desired result (3.1).

4. Epansion formula.

In this section, we establish an expansion formula in terms of a series of product of multivariable Gimel-function and generalized Legendre's associated function.

Theorem 2.

$$(1-x)^{\rho-\frac{u}{2}}(1+x)^{\sigma+\frac{v}{2}}\gimel\left((1-x)^{h_1}(1+x)^{k_1}z_1,\cdots,(1-x)^{h_r}(1+x)^{k_r}z_r\right)=$$

$$2^{\rho+\sigma}\sum_{R=0}^{\infty}\sum_{t=0}^{R}\frac{(2R-u+v+1)\Gamma(R-u+1)\Gamma(R+1-u+v+t)}{R!\Gamma(R+v+1)\Gamma(1-u+t)t!}(-R)_tP^{u,v}_{R-\frac{u-v}{2}}(x)$$

$$\gimel^{U;0,n_r+2:V}_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} 2^{h_1+k_1}z_1 & \mathbb{A};\ (\text{u-}\rho-t;h_1,\cdots,h_r;1),(-\sigma-v;k_1,\cdots,k_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{h_r+k_r}z_r & \mathbb{B};\ \mathbf{B},(\text{-1-}\rho-\sigma-t+u-v;h_1+k_1,\cdots,h_r+k_r;1):B \end{array}\right) \tag{4.1}$$

under the same existence condtions that (3.1).

Proof

$$f(x)=(1-x)^{\rho-\frac{u}{2}}(1+x)^{\sigma+\frac{v}{2}}\gimel\left((1-x)^{h_1}(1+x)^{k_1}z_1,\cdots,(1-x)^{h_r}(1+x)^{k_r}z_r\right)=\sum_{R=0}^{\infty}C_RP^{u,v}_{R-\frac{u-v}{2}}(x) \tag{4.2}$$

The above equation is valid since $f(u)$ is continuous and of bounded variation in the interval $(-1,1)$. Multiplying both sides of the above equation by $P^{u,v}_{l-\frac{u-v}{2}}(x)$ and integrating with respect to x between -1 to 1. Evaluating the left hand side with the help of the theorem 1 and the right hand side. Interchanging the order of summation, using ([3], p. 176, Eq. (75)) and then applying orthogonality property of the generalized Legendre's associated function with the help of the theorem 2, we get

$$C_l = \frac{2^{\rho+\sigma}(2l-u+v-1)\Gamma(l-u+1)}{l!\Gamma(l+v+1)} \sum_{t=0}^{l} \frac{(-l)_t\Gamma(1+l+v-u+t)}{t!\Gamma(1-u+t)}$$

$$\gimel^{U;0,n_r+2:V}_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} 2^{h_1+k_1}z_1 & \mathbb{A};\,(\text{u-}\rho-t;h_1,\cdots,h_r;1),(-\sigma-v;k_1,\cdots,k_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{h_r+k_r}z_r & \mathbb{B};\,\mathbf{B},(\text{-1-}\rho-\sigma-t+u-v;h_1+k_1,\cdots,h_r+k_r;1):B \end{array}\right) \tag{4.3}$$

Now subtituting the value of C_l in (4.2), we get the desired result.

4. Conclusion.

The main expansion formula (4.1) established here is unified and act as key formula. Thus the multivariable Gimel-function occurring in this expansion formula can be suitably specialized to a remarkably wide variety of special functions (or product of several such special functions) which are expressible in terms of E, F, G, H, I, Aleph-functions of one or more variables. Again the generalized Legendre's associated function involved in this paper reduces to a large number of special functions and others, therefore, from the expansion formula, we can further obtain various formulae involving a number of simpler functions.

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