A Study on FDM of Hyperbolic PDE in Comparative of Lax-Wendroff, Upwind, Leapfrog Methods on Numerical Analysis

Dr.S.Karunanithi[#1], M.Malarvizhi[#2], N.gajalakshmi[#3] , M.lavanya[*4]

*[#]Assistant Professor , *Research Scholar ,Thiruvalluvar University,Vellore*
PG & Research Department of Mathematics,Govt.Thirumagal Mills College,Gudiyattam,Vellore Dist,Tamilnadu,India-632602

Abstract *:- Three numerical method have been used to solve the one dimensional one way wave equation and second-order linear wave equation with constant coefficients. We discuss finite difference method for hyperbolic PDE. we consider the lax-wendroff scheme, the leapfrog scheme, upwind scheme finite difference scheme. We solve a one dimensional numerical experiment with specified initial and boundary condition, for which the exact solution is known using all these three schemes using some different values for the space and time step sizes denoted by h and k, respectively.*

Keywords :— *Linear equation, finite difference, one dimensional wave equation, lax-Wendorff scheme, leapfrog scheme, upwind scheme, error analysis, D'alembert solution and separation of variables*.

1.INTRODUCTION

A partial difference equation is an equation which contains its partial derivatives involving two or more independent variables. There are a large number of examples of partial differential equation in mathematical modeling, such as lax-wendroff scheme, leapfrog scheme, in second order wave equation, upwind scheme in one way wave equation.A differential equation involving more than one independent variable and its partial derivatives with respect to those variables is called a partial differential equation (PDE).

$$\frac{\partial}{\partial x}u(x,y) = 0$$

This equation implies that the function u(x, y) is independent of x. hence the general solution of this equation is u(x, y) =f(y), where f is an arbitrary function of y. the analogous ordinary differential equation is $\frac{dy}{dx} = 0$

Is general solution u(x) =c, where c is a constant. This example illustrates that general solutions of ODEs involve arbitrary constants, whereas solutions of PDEs involve arbitrary functions.

In general, one can classify PDEs with respect to different criterion,e.g.:

Order, dimension, linearity, initial /boundary value problem, etc.

By order of PDE we will understand the order of the highest derivative that occurs. A PDE is said to be linear if it is linear in unknown functions and their derivatives, with coefficients depending on the independent variables. The independent variables typically include one or more space dimensions and sometimes time dimension as well.

Consider the One way wave equation

$$u_t + a\,u_x = 0, \quad 0 < x < 1, \tag{1.1}$$
$$u(x,0) = u_0(x), \quad \text{IC},$$
$$u(0,t) = g_1(t) \quad \text{if } a \geq 0 \quad \text{or} \quad u(1,t) = g_r(t) \quad \text{if } a \leq 0.$$

Here g_1(t) and g_r(t) are prescribed boundary conditions from the left and right, respectively.

Second order linear wave equation

$$u_{tt} = a\,u_{xx}, \quad 0 < x < 1$$

$$u(x,0) = u_0(x), \quad u_t(x,0)= v_0(x), \quad \text{IC}, \tag{1.2}$$
$$u(0,t) = g_1(t), \quad u(1,t)= g_r(t), \quad \text{BC}.$$

in order to numerically approximating the solution of (1.1) and (1.2), we first discretize the x-t plane: set h=Δx(mesh width) and k=Δt(time step). This generates a lattice in the x-t plane, i.e., equally spaced mesh point (x_j , t^n) where x_j =jh, j=…,-1,0,1,…, and t^n =nk, n=0,1,… with these notations, we have, $x_{i+1/2}$ = $x_i + h/2$=(j+1/2)h. we denote the pointwise value of the exact solution to (1.1) and (1.2) at the grid points, (x_j,t^n), as u_j^n, and by v_j^n we denote an approximation of u_j^n.

We introduced the first and second order wave equation, method of characteristics, D'alembert solution and separation of variables and exact solution of the wave equation. We discussed about our numerical experiments and results.

2. PROBLEMS FORMULATION

First order wave equation

The first order wave equation is (c>0)

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \tag{1.3}$$
$$u(x,0) = u_0(x).$$

wave equation in one dimensional

one dimensional wave equation is

$$\frac{\partial 2u}{\partial 2t} = c^2 \frac{\partial 2u}{\partial 2x} \tag{1.4}$$

Initial condition:

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0)= v_0(x)$$

formally ,we can write Laplace equation as:

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) = \left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}\right) u(x,t).$$

D'ALEMBERT'S SOLUTION

$$u(x,t)=\frac{1}{2}\left\{u_0\,(x+ct) + u_0\,(x-ct) + \int_{x-ct}^{x+ct} v_0\,(s)ds\right\} \tag{1.5}$$

3.UPWIND SCHEME

we will start making use of the finite difference techniques to derive a discrete representation of equation (1.1) by first considering the derivative in the time. Taylor expanding the solution around (x_j,t^n) we obtain

$$u(x_j,t^n+\Delta t) = u(x_j,t^n) + \frac{\partial u}{\partial t}(x_j,t^n)\,\Delta t + o(\Delta t^2), \tag{1.6}$$

or, equivalently,

$$u_j^{n+1} = u_j^n + \frac{\partial u}{\partial t}\Big|_j \Delta t + o(\Delta t^2) \tag{1.7}$$

Isolating the time derivative and dividing by Δt we obtain

$$\frac{\partial u}{\partial t}\Big|_j = \frac{u_j^{n+1}-u_j^n}{\Delta t} + o(\Delta t^2) \tag{1.8}$$

Adopting a standard convenction, we will consider the finite difference respresentation of an m-th order differential operator $\partial^m u/\partial x^m$ in the generic x-direction (where x could either be a time or a spatial coordinate) to be of order p if and only if

$$\partial^m u/\partial x^m = l_\Delta(u) + o(\Delta t^p). \tag{1.9}$$

In way similar, the approximation (1.8) for the time derivative ,we can derive a first order, finite difference approximation to the space derivative as

$$\frac{\partial u}{\partial t}\Big|_j = \frac{u_j^n-u_{j-1}^n}{\Delta x} + o(\Delta x). \tag{2.0}$$

While formally similar , the approximation (2.0) suffers of the ambiguity, not present in expression(1.8), that the first order term in the taylor expansion can be equally expressed in terms of u_j^{n+1} and u_j^n

$$\frac{\partial u}{\partial t}\Big|_j = \frac{u_{j+1}^n - u_j^n}{\Delta x} + o(\Delta x)$$

This ambiguity is the consequenece of the first order approximation which prevents a proper "centring" of the finite difference stencil. However, and as long as we are concerned with an advection equation, this ambiguity is easily solved if we think that te differential equation will simply translate each point in the initial solution to the new position x+ vΔt over a time interval Δt. in this case, it is natural to select the points in the solution at the time level n that are "upwind" of the solution at the position j and at the time level n+1, as these are the ones causally connected with u_j^{n+1}.

Depending then on the direction in which the solution is translate , and hence on the value of the one way wave equation velocity v, two different finite difference representations can be given of equation (1.3) and these are

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -v\left(\frac{u_j^n - u_{j-1}^n}{\Delta x}\right) + o(\Delta t, \Delta x), \qquad \text{if } v > 0,$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -v\left(\frac{u_{j+1}^n - u_j^n}{\Delta x}\right) + o(\Delta t, \Delta x), \qquad \text{if } v < 0,$$

Respectively. As a result, the final finite difference algorithms for determing the solution at the new time level will have the form

$$u_j^{n+1} = u_j^n - \frac{v\Delta t}{\Delta x}(u_j^n - u_{j-1}^n) + o(\Delta t^2, \Delta x \Delta t), \qquad \text{if } v > 0, \qquad (2.1)$$

$$u_j^{n+1} = u_j^n - \frac{v\Delta t}{\Delta x}(u_{j+1}^n - u_j^n) + o(\Delta t^2, \Delta x \Delta t), \qquad \text{if } v < 0. \qquad (2.2)$$

4. LAX WENDROFF SCHEME

The lax Wendorff scheme is the second order accurate extension of the lax Fredrich's scheme. As for the case of the leapfrog scheme, in this case too we need two-time levels to obtain the solution at the new time level.

There are a number of different ways of deriving the lax Wendorff scheme but it is probably useful to look at is as to a combination of the lax Fredrich's scheme and of the leapfrog scheme.

Given the initial condition u(x,0)=f(x) and u_t9x,0)=g(x) for the wave equation show how to obtain starting values u_{n0} and u_{n1} for the difference methods .

The guiding principle here is that the starting values should represent the initial data with an error no worse then the local truncation error of the difference method, which in the present case is o(k^2+h^2). Obviously, then, we take u_{n0}=f(x_n), as this incurs error zero.

To decide on u_{n1}, let us suppose that f is in c^2 and that holds at t=0. Then Taylor's theorem gives

$$u(x_n, t_1) = u(x_n,0) + k u_t(x_n,0) + \frac{k^2}{2} u_n(x_n,0) + o(k^3)$$

$$= u(x_n,0) + k g(x_n) + \frac{k^2}{2} c^2 f''(x_n) + o(k^3)$$

$$= u(x_n,0) + k g(x_n) + \frac{k^2 c^2}{2h^2}\left[f(x_{n-1}) - 2f(x_n) + f(x_{n+1})\right] + o(k^2 h^2 + k^3)$$

Where, in the last step, f″(x_n) has been approximated through a second difference, according to the relation

$$g(x_n) = \frac{u_{n1} - u_{n0}}{k} - \frac{kc^2}{2h^2}\left[f(x_{n-1}) - 2f(x_n) + f(x_{n+1})\right]$$

is satisfied by the exact solution u to within o(kh^2+k^2). u_{n1} results in an error of higher order than o(k^2+h^2).

Lax wendroff scheme

$$u_j^{n+1} = u_j^n + \frac{k^2 a^2}{2h^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

5. LEAPFROG SCHEME

Both the forward time centred space and lax Fredrich are "one level" schemes with First order approximation for the time derivative and a second order approximation for the spatial derivative. In those circumstances yΔt should be taken significantly smaller than Δx (to achieve the desired accuracy), well below the limit imposed by the courant condition.

Second order accuracy in time can be obtained if we insert

$$\frac{\partial u}{\partial t}\Big|_j = \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + o(\Delta t^2),$$

In the forward time centred space, to find the leapfrog scheme

$$u_j^{n+1} = u_j^{n-1} - \frac{ak}{h}(u_{j+1}^n - u_{j-1}^n) + o(\Delta x^2).$$

The leapfrog scheme for $u_t = u_x$ is obtained by approximating both derivatives with a centred approximation,

$$u_j^{n+1} = u_j^{n-1} - \frac{ak}{h}(u_{j+1}^n - u_{j-1}^n).$$

6.NUMERICAL EXAMPLES

In this section we consider two numerical examples to prove which numerical method converge faster to analytical solution. Numerical result and error are the outcomes are represented by graphically.

Example-1

Approximated the solution to the hyperbolic problem

u_{tt} - 4 u_{xx} =0, 0 < x <1, 0 < t,

With boundary condition u(0,t) = u(1,t) =0, for 0 < t,

and initial conditions u (x,0) = sin ($2\pi x$), $0 \leq x \leq 1$, and u_t(x,0) = 0, $0 \leq x \leq 1$,

using h=0.1 and k=0.01.compare the results with the exact solution u(x ,t) = sin$2\pi x$cos4πt.

The approximate results and maximum errors are obtained and shown in tables-1(a,b) and the graphs of the numerical solutions are displayed in figures:1(a,b)

Tale -1 (a)

compare the results with the exact solution u(x ,t) = sin$2\pi x$cos4πt. The values for each x value lax wendroff method, upwind method, leapfrog method.**(t=1.0)**

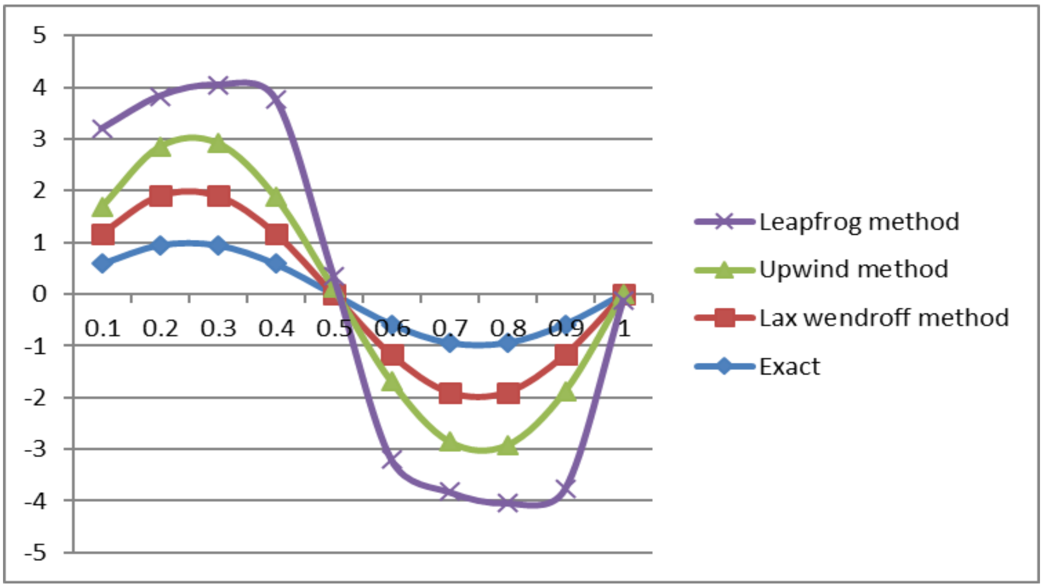
X value	Exact	Laxwendroff method	Upwind method	Leapfrog method
0.1	0.587785	0.583295	0.515131	1.511090
0.2	0.951057	0.943791	0.951057	0.978808
0.3	0.951057	0.943791	1.023711	1.124116
0.4	0.587785	0.583295	0.705342	1.891513
0.5	0	0	0.117557	0.235114
0.6	-0.587785	-0.583295	-0.515131	-1.511090
0.7	-0.951057	-0.943791	-0.951057	-0.978808
0.8	-0.951057	-0.943791	-1.023711	-1.124116
0.9	-0.587785	-0.583295	-0.705342	-1.891513
1.0	0	-0.011755	0	-0.117557

Table-1(b)

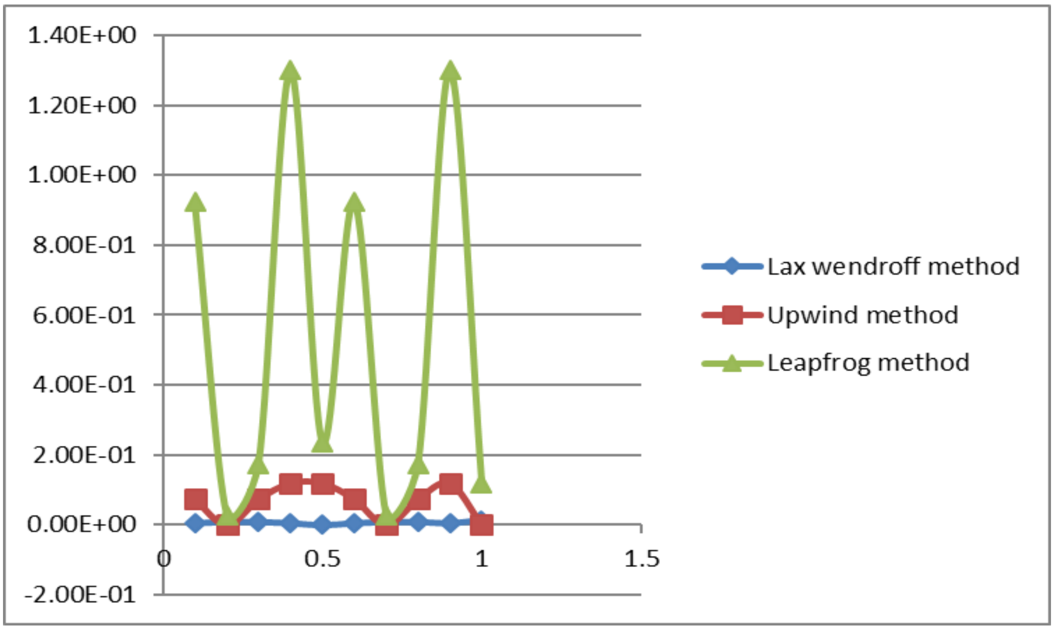
Shows the errors of lax wendroff method, upwind method, leapfrog method with exact method. These error values for each x are in the methods.

x-value	Lax wendroff method	Upwind method	Leapfrog method
0.1	4.49E-03	7.27E-02	9.23E-01
0.2	7.27E-03	0	2.78E-02
0.3	7.27E-03	7.27E-02	1.73E-01
0.4	4.49E-03	1.18E-01	1.30E+00
0.5	0	1.18E-01	2.35E-01
0.6	4.49E-03	7.27E-02	9.23E-01
0.7	7.27E-03	0	2.78E-02
0.8	7.27E-03	7.27E-02	1.73E-01
0.9	4.49E-03	1.18E-01	1.30E+00
1.0	1.18E-02	0	1.18E-01

Exact table



Error table



Example-2

Approximated the solution to the hyperbolic problem

u_{tt} - u_{xx} =0,   0 < x <1,   0 < t,

With boundary condition   u(0,t) = u(1,t) =0,   for 0 < t,

and initial conditions   u (x,0) = sin ($2\pi x$), $0 \le x \le 1$,   and u_t(x,0) = 0, $0 \le x \le 1$,

using h=0.1 and k=0.01.compare the results with the exact solution u(x ,t) = sin$2\pi x$cos$2\pi t$.

The approximate results and maximum errors are obtained and shown in tables-1(a,b) and the graphs of the numerical solutions are displayed in figures:2(a,b)

Table-1(a)

compare the results with the exact solution u(x ,t) = sin$2\pi x$cos$2\pi t$. The values for each x value lax wendroff method, upwind method, leapfrog method

(t=1.0)

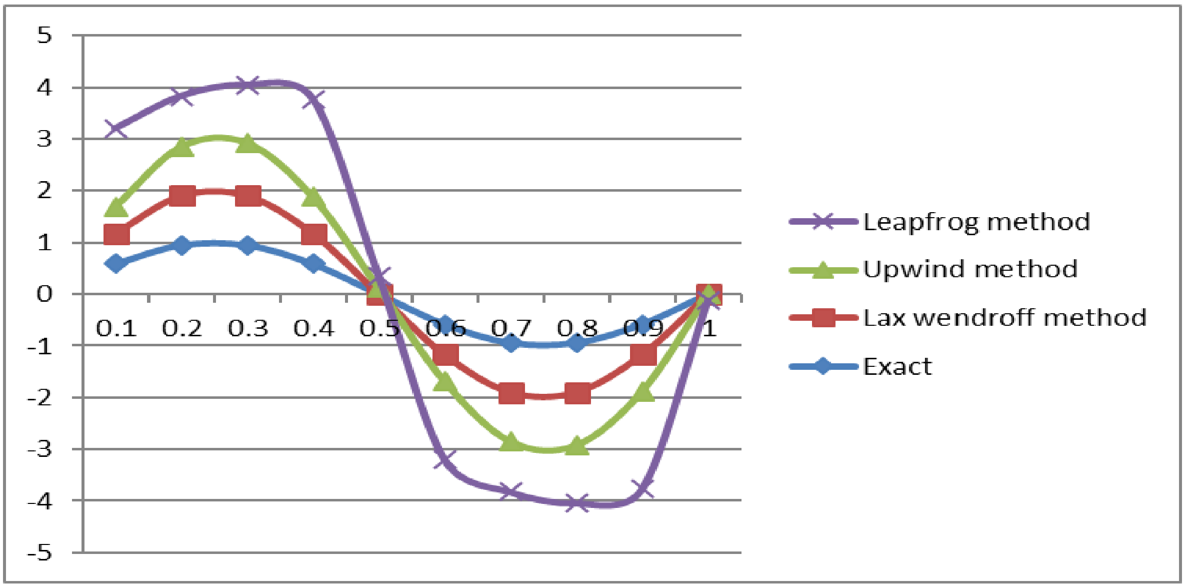
x-value	Exact	Lax wendroff method	Upwind method	Leapfrog method
0.1	0.5877852	0.5866627	0.5514581	1.6061960
0.2	0.9510565	0.9492402	0.9510565	1.0151351
0.3	0.9510565	0.9492402	0.9873836	1.0877894
0.4	0.5877852	0.5866627	0.6465638	1.7964073
0.5	0	0	0.0587785	0.1175571
0.6	-0.5877852	-0.5866627	-0.5514581	-1.6061960
0.7	-0.9510565	-0.9492402	-0.9510565	-1.0151351
0.8	-0.9510565	-0.9492402	-0.9873836	-1.0877894
0.9	-0.5877852	-0.5866627	-0.6465638	-1.7964073
1.0	0	-0.0029389	0	-0.0587853

Table-1(b)

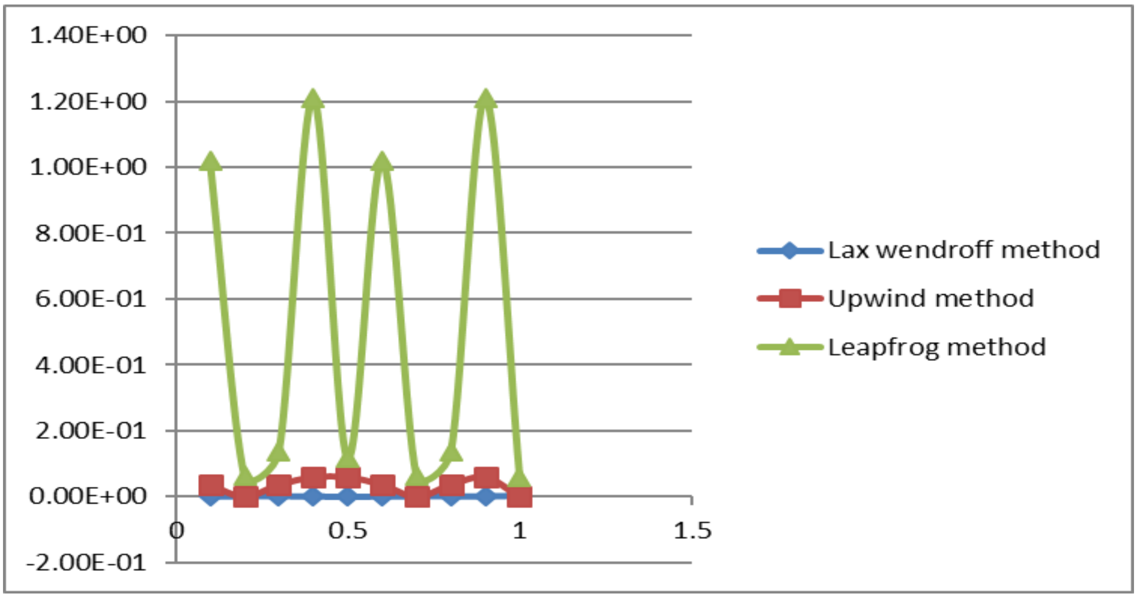
Shows the errors of lax wendroff method, upwind method, leapfrog method with exact method. These error values for each x are in the methods.

x-value	Lax wendroff method	Upwind method	Leapfrog method
0.1	1.12E-03	3.63E-02	1.02E+00
0.2	1.82E-03	0	6.41E-02
0.3	1.82E-03	3.63E-02	1.37E-01
0.4	1.12E-03	5.88E-02	1.21E+00
0.5	0	5.88E-02	1.18E-01
0.6	1.12E-03	3.63E-02	1.02E+00
0.7	1.82E-03	0	6.41E-02
0.8	1.82E-03	3.63E-02	1.37E-01
0.9	1.12E-03	5.88E-02	1.21E+00
1.0	2.94E-03	0	5.88E-02

Exact table



Error table



CONCLUSION

In this paper, three numerical methods have been used to solve a one dimensional one way wave equation and second order wave equation with specified initial and boundary conditions. When the one dimensional linear one way wave equation and second order wave equation is approximated by a numerical method, lax wendroff method, upwind method, leapfrog method for solving partial differential equation. The numerical test problems have shown that the numerical solution obtained by lax wendroff method are good agreement with exact solution. And errors are also calculated to further strengthen.In comparison test, the lax wendroff method has a better solution with specified initial and boundary condition problems comparing against upwind method & leapfrog method.

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