A Comparative Analysis of DTM and HAM Solutions for Hunter-Saxton Equation and Fisher Equation

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Abstract**—***In this article, two partial differential equations (PDEs), namely Hunter-Saxton equation and Fisher equation, which are non-linear in nature and arise in different physical situations are adopted and their solutions compared by the differential transform method (DTM) and homotopy analysis method (HAM). The comparative analysis outlines the significant features and effectiveness of these methods to the nonlinear PDEs, as it is possible to find the closed form like approximate solutions with high degree of accuracy when compared to the exact solutions. Also, for the problems under consideration it has been observed that the HAM results better as compared to DTM. Beside both the methods discussed, can be extended to solve a class of problems arising in different practical situations.*

Keywords—*Nonlinear PDEs, Differential Transform Method (DTM), Homotopy Analysis Method (HAM), Hunter-Saxton Equation, Fisher Equation.*

I. INTRODUCTION

The Hunter-Saxton equation was first introduced [1] as a model of the dynamics of a nematic liquid crystal. Liquid crystals consist of long rigid molecules in fluid phase and each molecule has an orientation [2], which is described in terms of the field of unit vectors $\left(\cos\left(u(x,t)\right), \sin\left(u(x,t)\right)\right)$ [3], where $x$is considered as space variable in a reference frame moving with an unperturbed wave speed and $t$is the time variable. The Hunter-Saxton equation also describes many other different physical situations as the high frequency limit of the Camassa-Holm equation [4], an integrable model equation for shallow water waves [5]. It also arises in the physical context for describing the geodesic flow on the diffeomorphism group of the circle (see [4, 6]), pseudo-spherical surfaces (see [7, 8]). It also possesses a completely integrable bi-Hamiltonian structure (see [9, 3]). The equation of the form (15) was first introduced [10] as a model for the propagation of a mutant gene [11]. Later a wide range of applications of Fisher equation has been found out in the fields of logistic population growth (see e.g., [11]), flame propagation [12], branching Brownian processes [13], neurophysiology [14], autocatalytic chemical reactions [13] and also in nuclear reactor theory [15]. In past few years, significant studies are made to find out numerical solutions for both of the Hunter-Saxton equation [16] and Fisher equation [17].

In the present study, a comparative analysis of solution methods namely DTM along with a wave transformation and HAM is being introduced to solve equations (1) and (15) subject to the initial conditions (2) and (16) respectively.

II. APPLICATION OF THE METHODS

This section consists of the solutions of the problems under consideration by applying the DTM and the HAM.

A. The Hunter-Saxton Equation

$$u_{xxt} + 2u_x u_{xx} + u u_{xxx} = 0 \tag{1}$$

With initial condition

$$u(x,0) = (x+1)^{\frac{2}{3}} + c \tag{2}$$

Where the solution in exact form is given as

$$u(x,t) = (x - ct + 1)^{\frac{2}{3}} + c \tag{3}$$

*1) **The DTM Solution:***Now using the wave variable $\eta = x - ct$ and considering $u(x,t) = v(\eta)$, the equation (1) with a bit manipulation is converted to the ODE as,

$$-cv''(\eta)+\frac{1}{2}\left(v'(\eta)\right)^2+v(\eta)v''(\eta)=0 \tag{4}$$

With the initial condition

$$v(0)=c+1 \tag{5}$$

We apply differential transform to equation (4) and the following [18], we get a recursive expression as:

$$V(m+2)=\frac{1}{c(m+1)(m+2)}\left\{\sum_{l=0}^{m}(l+1)(l+2)V(m-l)+\frac{1}{2}\sum_{l=0}^{m}(l+1)(m-l+1)V(l+1)V(m-l+1)\right\} \tag{6}$$

Where $V(m)$is the transformed function of $v(\eta)$under DTM. Also eq. (5) is transformed into:

$$V(0)=v(0)=c+1;\ V(1)=v'(0)=\alpha\ (\text{Say}) \tag{7}$$

Where $\alpha\in\mathbb{R}$ is a constant.

Now we derive the expressions for $V(m)$ for successive values of $m=0,1,\ldots,12$ with the help of recursive relation (6) and equations in (7). On using these expressions of $V(m)$ and by setting the range of x as $-1\le x\le 1$, with initial condition (2), the solution (approximate) of the Eq. (1) is finally computed in closed form like expression as:

$$u_{apprx}(x,t)= 1+c+0.66875(x-ct)-0.11180664062499998(x-ct)^2+0.04984712727864582(x-ct)^3-0.2916835807164509(x-ct)^4+0.01950633946041265(x-ct)^5-0.014131936556996873(x-ct)^6+0.010800837225704752(x-ct)^7-0.008577383624944437(x-ct)^8+0.007010819810110833(x-ct)^9-0.005818329709553986(x-ct)^{10}+0.004761991394440524(x-ct)^{11}-0.0041661302122412616(x-ct)^{12}. \tag{8}$$

*2) **The Ham Solution:***To start with the initial approximation is chosen as $u_0(x,t)=(1+x)^{\frac{2}{3}}+c$ and define the non-linear operator as:

$$N[\emptyset(x,t;\varrho)]=\frac{\partial^3\emptyset(x,t;\varrho)}{\partial x^2\partial t}+2\frac{\partial\emptyset(x,t;\varrho)}{\partial x}\frac{\partial^2\emptyset(x,t;\varrho)}{\partial x^2}+\emptyset(x,t;\varrho)\frac{\partial^3\emptyset(x,t;\varrho)}{\partial x^3} \tag{9}$$

Also the linear operator

$$L[\emptyset(x,t;\varrho)]=\frac{\partial\emptyset(x,t;\varrho)}{\partial t} \tag{10}$$

Satisfying the property: $L\left(a_1(x)\right)=0$ (11)

Where $a(x)$ is the constant of integration, so that $L^{-1}=\int_0^t(.)dt$

Considering, $\Re_k(\overrightarrow{u_{k-1}})=\frac{\partial^3 u_{k-1}(x,t)}{\partial x^2\partial t}+2\frac{\partial u_{k-1}(x,t)}{\partial x}\frac{\partial^2 u_{k-1}(x,t)}{\partial x^2}+u_{k-1}(x,t)\frac{\partial^3 u_{k-1}(x,t)}{\partial x^3}$ (12)

The solution of the k^{th}-order deformation equation, for $k\ge 1$ is written as:

$$u_k(x,t)=\chi_k u_{k-1}(x,t)+\hbar L^{-1}[H(x,t)\Re_k(\overrightarrow{u_{k-1}})] \tag{13}$$

Since $k\ge 1$ and$\chi_k=1$, we set$\hbar=-1$, $H(x,t)=1$ and the terms $u_1(x,t),u_2(x,t),u_3(x,t),\ldots$ can be obtained successively.

Finally the closed form like solution can be expressed as: $u(x,t)=u_0(x,t)+\sum_{k=1}^{\infty}u_k(x,t)$ (14)

B. The Fisher Equation

$$u_t=u_{xx}+6u(1-u) \tag{15}$$

With initial condition

$$u(x,0)=\frac{1}{(1+e^x)^2} \tag{16}$$

Where the solution in exact form is given as

$$u(x,0)=\frac{1}{(1+e^{x-5t})^2} \tag{17}$$

*1) **The DTM Solution:**Using the wave variable $\eta=x-ct$ and letting $u(x,t)=v(\eta)$, the Eq. (15) is converted to the following ODE*

$$v''(\eta) + cv'(\eta) + 6v(\eta) - 6v^2(\eta) = 0 \quad (18)$$

With the initial condition

$$v(0) = \frac{1}{4} \quad (19)$$

Now we apply differential transform to Eq. (18) and (19), and following [18], we get the recursive relation as:

$$V(m+2) = \frac{6}{(m+1)(m+2)}\{\sum_{l=0}^{m} V(l)V(m-l) - V(m)\} - \frac{c}{(m+2)}V(m+1) \quad (20)$$

Also using the initial condition (19)

$$V(0) = v(0) = \frac{1}{4};\ V(1) = v'(0) = \beta\ \text{(Say)} \quad (21)$$

Where $\beta \in \mathbb{R}$ is a constant.

Now we successively derive the expressions for $V(m)$ for values of $m = 0,1,\dots,8$ with the help of recursive Eq. (20) and Eq. in (21). On considering the expressions of $V(m)$, and setting the range of x as $-1 \le x \le 1$, in view of the initial condition (16), the solution (approximate) of the Eq. (15) is finally given by the following closed form like solution as:

$$u_{apprx}(x,t) = 0.25 - 0.2341778830122118(x - 5.26002\,t) + 0.05338806650000005(x - 5.26002\,t)^2 + 0.023481829317394842(x - 5.26002\,t)^3 - 0.030515813695268568(x - 5.26002\,t)^4 + 0.012430551363650203 3(x - 5.26002\,t)^5 - 0.010846367727691264(x - 5.26002\,t)^6 + 0.008745017590762794(x - 5.26002\,t)^7 - 0.00017590920303017787(x - 5.26002\,t)^8 \quad (22)$$

2) ***The Ham Solution:***To start with the initial approximation is chosen as $u_0(x,t) = \frac{1}{(1+e^x)^2}$ and define the non-linear operator as:

$$N[\emptyset(x,t;\varrho)] = \frac{\partial^3\emptyset(x,t;\varrho)}{\partial x^2\partial t} + 2\frac{\partial\emptyset(x,t;\varrho)}{\partial x}\frac{\partial^2\emptyset(x,t;\varrho)}{\partial x^2} + \emptyset(x,t;\varrho)\frac{\partial^3\emptyset(x,t;\varrho)}{\partial x^3} \quad (23)$$

Also the linear operator: $L[\emptyset(x,t;\varrho)] = \frac{\partial\emptyset(x,t;\varrho)}{\partial t}$ (24)

Satisfying the property: $L\big(a_2(x)\big) = 0$ (25)

Where $a_2(x)$ is the constant of integration, so that $L^{-1} = \int_0^t(.)dt$

Considering, $\Re_k(\overrightarrow{u_{k-1}}) = \frac{\partial^3 u_{k-1}(x,t)}{\partial x^2\partial t} + 2\frac{\partial u_{k-1}(x,t)}{\partial x}\frac{\partial^2 u_{k-1}(x,t)}{\partial x^2} + u_{k-1}(x,t)\frac{\partial^3 u_{k-1}(x,t)}{\partial x^3}$ (26)

The solution of the k^{th}-order deformation equation, for $k \ge 1$ is written as per equation (13).

Since $K \ge 1$ and$\chi_K = 1$, we set$\hbar = -1$, $H(x,t) = 1$ and the terms $u_1(x,t), u_2(x,t), u_3(x,t)$, ...can be obtained successively.

Finally the closed form like solution is expressed in the form of Eq. (14).

III.DISCUSSION OF THE RESULTS

In this study, a comparative analysis is done for solution of two nonlinear time dependent PDEs namely Hunter-Saxton equation and Fisher equation by means of Differential transform method and Homotopy analysis method. In the procedure of DTM, the wave transformations are injected on the PDEs, to convert them into ODEs. Where as in HAM, the auxiliary parameter and the auxiliary function are taken as -1 and 1 respectively, which are well justified to the problems under consideration. To justify the accuracy of both the methods, numerical simulations has been made in terms of six figures and a table for each problem.

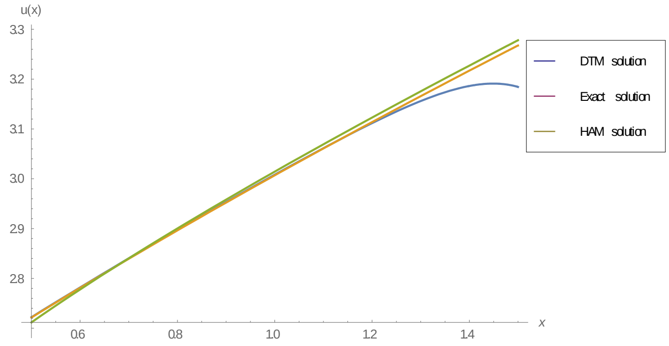

Fig. 1 Comparison graphs for DTM, HAM and Exact solutions of Eq. (1) for 0.5 ≤ x ≤ 1.5 and t = 0.1

In Fig. 1, a plot of the solutions of Eq. (1) is depicted to compare the solutions obtained by DTM and HAM to that of the exact analytical method, for $x \in [0.5,1.5]$ and $t = 0.1$. It has been observed that as time advances, the slope of the trajectories provided by HAM and the exact solutions maintain nearly a constant proportionality, whereas the slope of the trajectory of DTM solution shows a declination in nature for the range of values of $x = 1.3$ onwards and the extant of agreement with the exact solution gradually ceases.

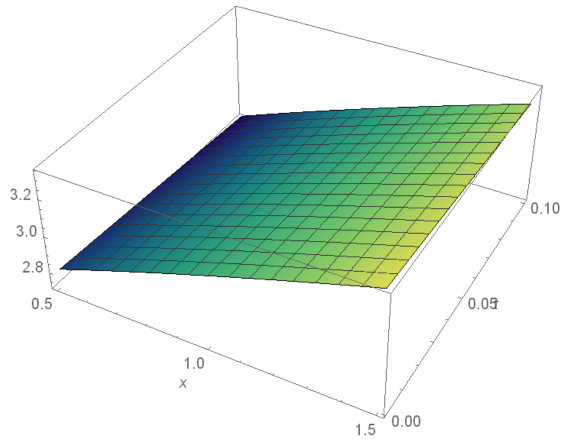

Fig. 2 Exact solution surface of Eq. (1) for 0.5 ≤ x ≤ 1.5 and 0.001 ≤ t ≤ 0.1

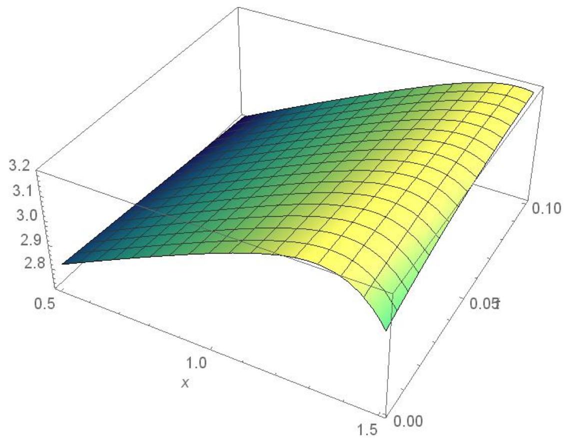

Fig. 3 Solution surface obtained by DTM of Eq. (1) for 0.5 ≤ x ≤ 1.5 and 0.001 ≤ t ≤ 0.1

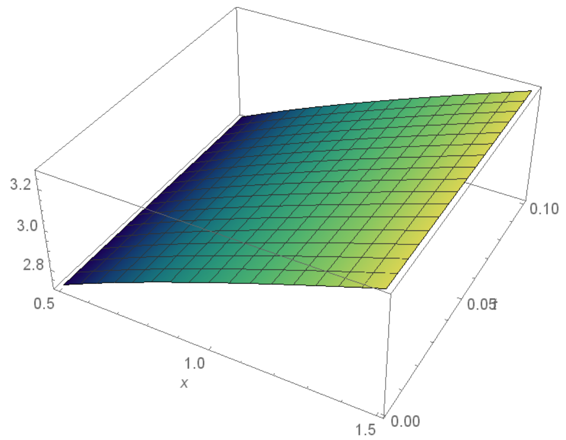

Fig. 4 Solution surface obtained by HAM of Eq. (1) for 0.5 ≤ x ≤ 1.5 and 0.001 ≤ t ≤ 0.1

Fig. 2, Fig. 3 and Fig. 4 demonstrate the solution surfaces of Eq. (1) as obtained by exact solution, DTM solution and HAM solution respectively for specified $x \in [0.5,1.5]$ and for $t \in [0.001,0.1]$. It can be seen very clearly that the exact solution surface and the HAM solution surface agree very closely for the specified range of x, whereas the surface of the DTM solution shows a concavity in nature from $x = 1.3$ onwards. It is also observed that the inclinations of all of the solution surfaces agree to one another for $x \in [0.5,1.3]$.

Table 1 Table of comparisons between results of present solutions and exact solution with error analysis for Eq. (1)

Values of x	**Values of t**	**Exact Solution**	**DTM Solution**	**HAM Solution**	**Absolute Error in DTM**	**Absolute Error in HAM**
0.55	0.01	2.83068	2.83164	2.82982	0.00096	0.00086
0.6	0.02	2.85083	2.85185	2.84979	0.00102	0.00104
0.65	0.03	2.87083	2.8719	2.87009	0.00107	0.00074
0.7	0.04	2.89069	2.89181	2.8906	0.00112	0.00009
0.75	0.05	2.9104	2.91157	2.91123	0.00117	0.00083
0.8	0.006	2.97479	2.97604	2.97493	0.00125	0.00014
0.85	0.007	3.0013	3.00249	3.00155	0.00119	0.00025
0.9	0.008	3.02757	3.0286	3.02794	0.00103	0.00037
0.95	0.009	3.05362	3.05428	3.05411	0.00066	0.00049
1.0	0.0001	3.08732	3.08693	3.08733	0.00039	0.00001

The above table describes a numerical comparison of the present solutions with the exact solution of Eq. (1). For different values of x & t the solutions obtained by DTM and HAM are found to be very much closer to that of the exact method. The absolute error for both the methods that are found in the table, are found to be very less, whereas the absolute error for HAM solution is much less as compared to the DTM solution.

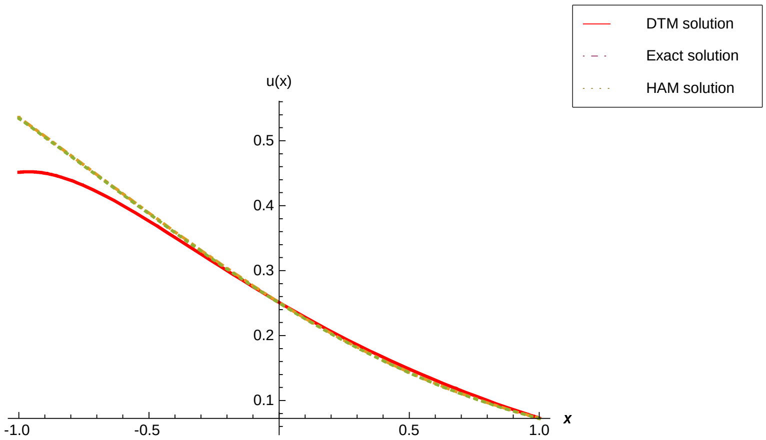

Fig. 5 Comparison graphs for solution of Eq. (15) for -1 ≤ x ≤ 1 and t = 0.001

In Fig. 5, a plot of the solutions of Eq. (15) is depicted to compare the solutions obtained by DTM and HAM to that of exact analytical solution, for $x \in [-1,1]$ and $t = 0.001$. It is observed from the figure that the DTM solution graph closely agrees with the exact solution graph roughly for $x \in [-0.4,1]$, whereas the solution graph of HAM replicates the solution graph of the exact method and is found to be in excellent agreement.

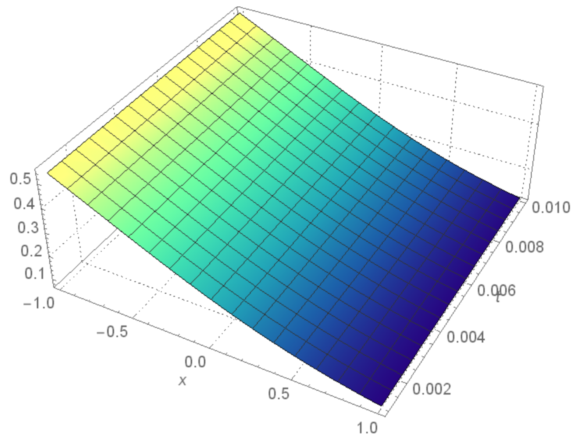

Fig. 6 Exact solution surface of Eq. (15) for -1 ≤ x ≤ 1 and 0.001 ≤ t ≤ 0.01

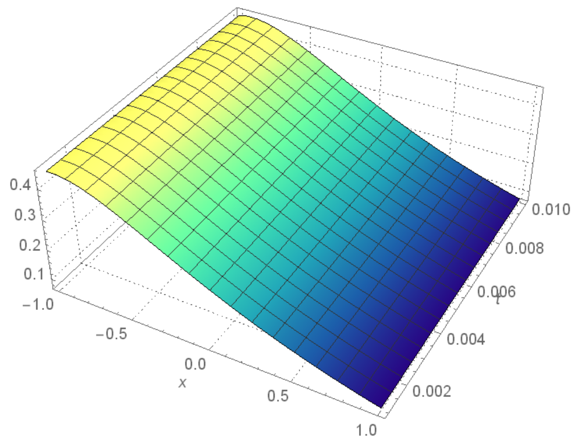

Fig. 7 Solution surface obtained by DTM of Eq. (15) for -1 ≤ x ≤ 1 and 0.001 ≤ t ≤ 0.01

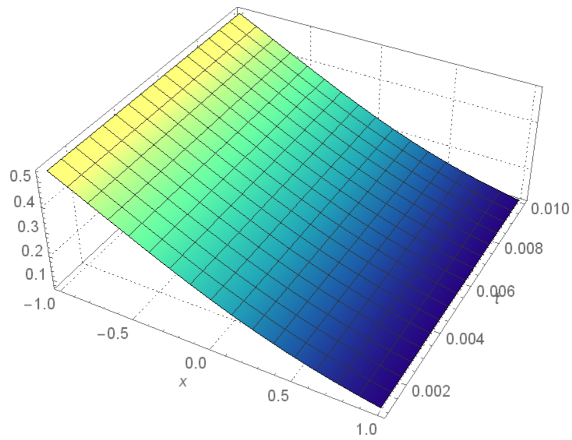

Fig. 8 Solution surface obtained by HAM of Eq. (15) for $-1 \leq x \leq 1$ and $0.001 \leq t \leq 0.01$

Figures 6, 7 and 8 show the solution surfaces of Eq. (15) as obtained by exact solution, DTM solution and HAM solution respectively for specific $x \in [-1,1]$ and for $t \in [0.001,0.01]$. For all the surfaces, the inclination of surface with the horizontal axis are almost similar and precisely for the solution surface obtained by HAM it seems to be more accurate to that of the exact solution as compared to the DTM solution surface.

Table 2 Table of comparisons between results of present solutions and exact solutions with error analysis for Eq. (15)

Values of x	Values of t	Exact Solution	DTM Solution	HAM Solution	Absolute Error in DTM	Absolute Error in HAM
0.3	0.001	0.182141	0.186042	0.181278	0.003901	0.000863
0.5	0.003	0.145216	0.150627	0.143034	0.005411	0.002182
0.65	0.005	0.121553	0.127394	0.118405	0.005841	0.003148
0.7	0.007	0.11534	0.121489	0.111136	0.006149	0.004204
0.9	0.009	0.089034	0.094379	0.084699	0.005345	0.004335
-0.9	0.002	0.508369	0.450813	0.505703	0.057556	0.002666
-0.8	0.004	0.481964	0.442173	0.476615	0.039791	0.005349
-0.7	0.006	0.455362	0.427831	0.447363	0.027531	0.007999
-0.6	0.008	0.428702	0.409597	0.418133	0.019105	0.010569
-0.5	0.009	0.400657	0.387463	0.388951	0.013194	0.011706

The above table elaborates a comparison of the present solutions with the exact solutions for Eq. (15). For different time periods and for different values of x, the above table clearly signifies the similarities of both the methods in terms of absolute errors but for HAM solution it is noticeably less as compared to DTM solution.

IV. CONCLUSIONS

A comparative analysis has been done to solve Hunter-Saxton equation and Fisher equation by applying DTM and HAM. In DTM, the PDEs are converted to their respective ODEs by a suitable wave transformation and are finally converted to algebraic equations by applying differential transform operator. In HAM, auxiliary parameter and function have been chosen as -1 and 1 respectively in accordance with the equations under consideration. Both the methods work very efficiently to deal with the problems and give closed form like approximate solutions. But to the extent of accurateness of the solutions and effort of computations, it has been observed in terms of figures and tables, that HAM results better as compared to DTM for the problems under consideration. The idea of the present article can also be extended to a class of nonlinear PDEs of even dimension more than one.

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