Some properties of harmonic convex and harmonic quasi-convex functions

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Abstract

In this paper, we introduce a new class of convex function which is known as harmonically convex function. It is shown that harmonically log-convex function implies that harmonically convex functions which implies that harmonically quasi-convex functions. Results proved in this paper may stimulate further research in this field.

Keywords: Convex functions, harmonically convex functions, harmonically quasi-convex functions, harmonically log-convex functions, harmonically pseudo-convex functions.

1 Introduction

Convex analysis plays a significant role in pure and applied mathematics especially in optimization theory and non linear programming due to its symmetry in shape and properties of convex sets and functions. Several new classes of convex sets and convex functions have been introduced and investigated, which make this area of research very attractive and useful. A significant class of convex functions, called harmonic convex was introduced by Anderson et al. [1] and Iscan [4]. Noor and Noor [6, 7] have shown that the optimality conditions of the differentiable harmonic convex functions on the harmonic convex set can be expressed by a class of variational inequalities, which is called the harmonic variational inequality. For recent developments and applications, see [5-7, 8-13].

To the best of my knowledge, this field is new one and has not been developed as yet. In this paper, we show that harmonic convex and harmonic quasi convex functions have some nice properties [2]. We obtained the necessary and sufficient characterization of a differentiable harmonic convex and harmonic quasi convex functions. The ideas and techniques used in this paper are very interesting and may inspire further research in this field. This is the main motivation of this article.

2 Preliminaries

In this section, we recall some basic results and define the concept of harmonically convex and harmonically quasi-convex functions.

Definition 2.1. A set $K \subseteq \mathbb{R}$ is said to be convex set, if $y + t(x - y) \in K, \quad \forall x,\ y \in K,\ \ t \in [0, 1]$.

Definition 2.2. A function $f \colon K \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex functions, if $f(y + t(x - y)) \leq (1 - t)f(x) + tf(y)$ $\forall x,\ y \in K,\ \ t \in [0, 1]$.

Definition 2.3. [8] A set $K \subset \mathbb{R}_+$ is said to be harmonic convex set, if $\frac{xy}{y+t(x-y)} \in K \ \ \forall\ x,\ y \in k,\ \ t \in [0, 1]$.

Definition 2.4. [8] A function $f \colon K \subset \mathbb{R}_+ \to \mathbb{R}$ is said to be harmonic convex function, if

$$f\left(\frac{xy}{y+t(x-y)}\right) \le (1-t)f(x) + tf(y), \forall\, x,\ y \in K,\ t \in [0,1].$$

Definition 2.5. The function f is said to be harmonic concave function iff $-f$ is harmonic convex function.

Definition 2.6. A function $f \colon K \subset \mathbb{R}_+ \to \mathbb{R}$ is said harmonic quasi-convex function, if

$$f\left(\frac{xy}{y+t(x-y)}\right) \le \max\{f(x),\ f(y)\}, \forall\, x,\ y \in K,\ t \in [0,1].$$

Definition 2.7. (a) The function f is said to be harmonic quasi-concave iff $-f$ is harmonic quasi-convex.
(b) A function f is harmonic quasi-convex, if whenever $f(y) \ge f(x)$.
(c) A function f is said to be strictly harmonic quasi-convex, if $f(y) > f(x)$.

Definition 2.8. [3] A function $f \colon K \subset \mathbb{R}_+ \to \mathbb{R}$ is said to be harmonic log-convex function, if

$$f\left(\frac{xy}{y+t(x-y)}\right) \le (f(x))^{1-t} \cdot (f(y))^{t}, \forall\, x,\ y \in K,\ t \in [0,1].$$

Definition 2.9. [2] Let K be a non-empty in $\mathbb{R}^n$ and $f \colon K \to \mathbb{R}$ be a function. Then epigraph of f denoted by $E(f)$ and is defined by $E(f) = \{(x,\ \lambda) : x \in K,\ \lambda \in \mathbb{R},\ f(x) \le \lambda\}$.

Definition 2.10. A function $f \colon K \to \mathbb{R}$ is said to be harmonic pseudo-convex function with respect to a strictly positive function $\eta(\cdot, \cdot)$ such that $f(x) > f(y)$

$$\Rightarrow f\left(\frac{xy}{y+t(x-y)}\right) < f(x) + t(t-1)\eta(x,y), \forall\, x,\ y \in K,\ t \in (0,1).$$

Definition 2.11. Let K be a non-empty in $\mathbb{R}_+$. Then the function $f \colon K \to \mathbb{R}$ is said to be
(a) harmonic pseudo-convex function, if $\forall\, x,\ y \in K$ with $\langle f'(y),\ \frac{xy}{y-x}\rangle \ge 0$, we have $f(x) \ge f(y)$.
(b) harmonic pseudo quasi convex function, if $\forall\, x,\ y \in K$ with $f(x) \le f(y)$, we have $\langle f'(y),\ \frac{xy}{y-x}\rangle \le 0$.

Theorem 2.12. *Let K be a harmonic convex set and $f \colon K \to \mathbb{R}$ be a harmonic convex function. Then any local minimum of f is a global minimum.*

Proof. Let $x \in K$ be a local minimum of a harmonic convex function f.
Suppose on the contrary that $f(y) < f(x),\ y \in K$, since f is harmonic convex function. Then,

$$f\left(\frac{xy}{y+t(x-y)}\right) \le (1-t)f(x) + tf(y) \ \ \forall\, x,\ y \in K,\ t \in [0,1]$$
$$\le f(x) - tf(x) + tf(y)$$
$$= f(x) + t(f(y) - f(x))$$
$$\Rightarrow f\left(\frac{xy}{y+t(x-y)}\right) - f(x) \le t\,[f(y) - f(x)].$$

For some $t > 0$, it follows that $f\left(\frac{xy}{y+t(x-y)}\right) < f(x)$, which is a contradiction.
Hence every local minimum of f is global minimum. ∎

Theorem 2.13. *If $f \colon K \subset \mathbb{R}_+ \to \mathbb{R}$ be a harmonic log convex function on K, then f is harmonic convex function implies f is harmonic quasi convex function.*

Proof. Suppose f is harmonic log convex function. Then for all $x,\ y \in K$ and $t \in [0,1]$,

$$f\left(\frac{xy}{y+t(x-y)}\right) \leq (f(x))^{1-t} \cdot (f(y))^t$$
$$\leq (f(x))^{1-t} + (f(y))^t$$
$$\leq (1-t)f(x) + tf(y)$$
$$\leq \max\{f(x),\ f(y)\}.$$

This proves that f is harmonic log convex function
$\Rightarrow f$ is harmonic convex function
$\Rightarrow f$ is harmonic quasi convex function.
The converse of the theorem (2.13) need not be true. ∎

3 Main Result

In this section, we discuss some properties of harmonic convex function and harmonic quasi convex function.

Theorem 3.1. *If K_1 and K_2 are two harmonic convex sets, then $K_1 \cap K_2$ is also harmonic convex set.*

Proof. Let $x,\ y \in K_1 \cap K_2,\ t \in [0,1]$. Then $x,\ y \in K_1 \cap K_2$
$\Rightarrow\quad x,\ y \in K_1$ and $x,\ k \in K_2$
$\Rightarrow\quad \frac{xy}{y+t(x-y)} \in K_1$ and $\frac{xy}{y+t(x-y)} \in K_2$
$\Rightarrow\quad \frac{xy}{y+t(x-y)} \in K_1 \cap K_2,\quad t \in [0,1]$
$\Rightarrow\quad K_1 \cap K_2$ is convex set. ∎

Theorem 3.2. *Let K be a harmonic convex set and $f \colon K \to \mathbb{R}$ be a harmonic convex function. Then $f = \lambda f$ is also harmonic convex function, where $\lambda \geq 0$.*

Proof. Let K be harmonic convex set. Then for $x,\ y \in K,\ t \in [0,1]$, we have

$$f\left(\frac{xy}{y+t(x-y)}\right) = \lambda f\left(\frac{xy}{y+t(x-y)}\right)$$
$$\leq \lambda\left[(1-t)f(x) + tf(y)\right]$$
$$= (1-t)\lambda f(x) + t\lambda f(y)$$
$$= (1-t)f(x) + tf(y)$$

$\Rightarrow\quad f = \lambda f$ is harmonic convex function. ∎

Theorem 3.3. *Let $f \colon K \subset \mathbb{R}_+ \to \mathbb{R}$ be a harmonic convex function on harmonic convex set K. Then the level set $K_\lambda = \{x \in K : f(x) \leq \lambda,\ \lambda \in \mathbb{R}\}$ is harmonic convex set.*

Proof. Let $x,\ y \in K_\lambda$. Then $f(x) \leq \lambda,\ f(y) \leq \lambda$.

$$Now\ \ f\left(\frac{xy}{y+t(x-y)}\right) \leq (1-t)f(x) + tf(y)$$
$$\leq (1-t)\lambda + t\lambda$$
$$= \lambda - t\lambda + t\lambda$$
$$= \lambda$$
$$\Rightarrow\ \ f\left(\frac{xy}{y+t(x-y)}\right) \leq \lambda\ \ \forall\ x,\ y \in K_\lambda$$

$\Rightarrow\quad K_\lambda$ is harmonic convex set. ∎

Theorem 3.4. *The function $f\colon K \subset \mathbb{R}_+ \to \mathbb{R}$ is harmonic convex iff $E(f)$ is harmonic convex set.*

Proof. First, suppose that f is harmonic convex function and let $(x,\ \lambda_1),\ (x,\ \lambda_2) \in E(f)$. Then $f(x) \le \lambda_1,\ \ f(y) \le \lambda_2$. For $t \in [0,1]$,

$$f\left(\frac{xy}{y+t(x-y)}\right) \le (1-t)f(x) + tf(y)$$
$$\le (1-t)\lambda_1 + t\lambda_2$$

$\Rightarrow \quad \left(\frac{xy}{y+t(x-y)},\ (1-t)\lambda_1 + t\lambda_2\right) \in E(f)$

$\Rightarrow \quad E(f)$ is harmonic convex set.

Conversely, suppose $E(f)$ is harmonic convex set and let $x,\ y \in K$.

Then $(x,\ f(x)),\ (y,\ f(y)) \in E(f)$, we have

$\left(\frac{xy}{y+t(x-y)},\ (1-t)f(x) + tf(y)\right) \in E(f)$,

$\Rightarrow \quad f\left(\frac{xy}{y+t(x-y)}\right) \le (1-t)f(x) + tf(y)$

$\Rightarrow \quad f$ is harmonic convex function. ∎

Theorem 3.5. *[8] Let f and g are two harmonically convex functions. If f and g are similarly ordered, then fg is also harmonically convex function.*

Proof. Let f and g are harmonically convex functions. Then

$$f\left(\frac{xy}{y+t(x-y)}\right)g\left(\frac{xy}{y+t(x-y)}\right) \le [(1-t)f(x) + tf(y)][(1-t)g(x) + tg(y)]$$
$$= (1-t)^2 f(x)g(x) + t(1-t)f(x)g(y) + t(1-t)f(y)g(x) + t^2 f(y)g(y)$$
$$= (1-t)f(x)g(x) + tf(y)g(y) + (1-t)^2 f(x)g(x)$$
$$+ t(1-t)[f(x)g(y) + f(y)g(x)] - (1-t)f(x)g(x) - tf(y)g(y) + t^2 f(y)g(y)$$
$$\le (1-t)f(x)g(x) + tf(y)g(y)$$

$\Rightarrow \quad f\left(\frac{xy}{y+t(x-y)}\right)g\left(\frac{xy}{y+t(x-y)}\right) \le (1-t)f(x)g(x) + tf(y)g(y).$

This proves that product of two harmonically convex functions is harmonically convex function. ∎

Theorem 3.6. *If $f\colon K \subset \mathbb{R}_+ \to \mathbb{R}$ be harmonic quasi convex function and $g\colon \mathbb{R} \to \mathbb{R}$ is increasing function, then $gof\colon K \to \mathbb{R}$ is harmonic quasi convex function.*

Proof. Suppose f is harmonic quasi convex function and g is increasing function. Then

$$(gof)\left(\frac{xy}{y+t(x-y)}\right) = g\left[f\left(\frac{xy}{y+t(x-y)}\right)\right]$$
$$\le g\left[\max\{f(x),\ f(y)\}\right]$$
$$= \max\{gof(x),\ gof(y)\}$$
$$= \max\{(gof)(x),\ (gof)(y)\}$$

$\Rightarrow \quad gof$ is harmonic quasi convex function. ∎

Theorem 3.7. *If $f\colon K \subset \mathbb{R}_+ \to \mathbb{R}$ is harmonic convex function such that $f(x) > f(y)$, then f is harmonic pseudo convex function with respect to strictly positive fiunction $\eta(\cdot,\ \cdot)$.*

Proof. Suppose $f(x) > f(y)$ and f is harmonic convex. Then

$$f\left(\frac{xy}{y+t(x-y)}\right) \le (1-t)f(x) + tf(y)$$

$$= f(x) - tf(x) + tf(y)$$
$$= f(x) + t[f(y) - f(x)]$$
$$< f(x) + t(1-t)\,[f(y) - f(x)]$$
$$= f(x) + t(t-1)\,[f(x) - f(y)]$$
$$< f(x) + t(t-1)\eta(y,\ x),$$

where $\eta(y,\ x) = f(x) - f(y) > 0$. ∎

Theorem 3.8. *Let $f \colon K \subset \mathbb{R}_+ \to \mathbb{R}$ be a differentiable on a non empty harmonic convex set K. Then f is harmonic quasi convex iff $f(x) \le f(y) \Rightarrow \ \langle f'(y),\ \frac{xy}{y-x}\rangle \le 0 \ \ \forall\ x,\ y \in K$.*

Proof. First, suppose f is harmonic quasi convex and $x,\ y \in K$ such that $f(x) \le f(y)$. Then by Taylor series,

$$f\left(\frac{xy}{y + t(x-y)}\right) = f(y) + t\langle f'(y),\ \frac{xy}{y-x}\rangle + t\|\frac{xy}{y-x}\|\ \alpha\left[y;\ t\left(\frac{xy}{y-x}\right)\right]. \tag{3.1}$$

As $t \to 0,\ \ \alpha\left[y;\ t\left(\frac{xy}{y-x}\right)\right] \to 0$

since f is harmonic quasi convex, $f\left(\frac{xy}{y+t(x-y)}\right) \le f(y)$.

Hence from equation (3.1), we get

$$t\langle f'(y),\ \frac{xy}{y-x}\rangle + t\|\frac{xy}{y-x}\|\ \alpha\left[y;\ t\left(\frac{xy}{y-x}\right)\right] \le 0$$

$$\Rightarrow \ \ \langle f'(y),\ \frac{xy}{y-x}\rangle + t\|\frac{xy}{y-x}\|\ \alpha\left[y;\ t\left(\frac{xy}{y-x}\right)\right] \le 0.$$

As $t \to 0$, we have
$\langle f'(y),\ \frac{xy}{y-x}\rangle \le 0$.
Conversely, suppose that $x,\ y \in K$ and $f(x) \le f(y)$.
We need to show that $f\left(\frac{xy}{y+t(x-y)}\right) \le f(y),\ \forall\ x,\ y \in K,\ t \in (0,1)$.
For this, we need to show that the set $L = \{x' : x' = \frac{xy}{y+t(x-y)},\ t \in (0,1),\ f(x') > f(y)\}$ is empty.
On contrary, suppose that $\exists\ x' \in L$
$x' = \frac{xy}{y+t(x-y)},\ t \in (0,1)$ and $f(x') > f(y)$
since f is differentiable
$\Rightarrow \ \ \ f$ is continuous and hence $\exists\ \delta \in (0,1)$ such that $f\left(\frac{x'y}{y+\mu(x-y)}\right) > f(y)$ for each $\mu \in (\delta,1)$ and
$f(x') > f\left(\frac{x'y}{y+\delta(x'-y)}\right)$.
By Mean value theorem, we have

$$0 < f(x') - f\left(\frac{x'y}{y+\delta(x'-y)}\right) = (1-t)\langle f'(\hat{x}),\ \frac{x'y}{y-x'}\rangle \tag{3.2}$$

where $\hat{x} = \frac{x'y}{y+\mu'(x'-y)}$ for some $\mu' \in (\delta,1)$.

$$\Rightarrow \ \ f(\hat{x}) > f(y). \tag{3.3}$$

From equation (3.2), we have

$$\langle f'(\hat{x}),\ \frac{x'y}{y-x'}\rangle = \frac{f(x') - f\left(\frac{x'y}{y+\delta(x'-y)}\right)}{(1-t)} > 0$$

$$\Rightarrow \ \ \langle f'(\hat{x}),\ \frac{x'y}{y-x'}\rangle > 0. \tag{3.4}$$

But from equation (3.3), we have

$f(\hat{x}) > f(y) \geq f(x)$ and $\hat{x}$ is harmonic combination of x and y.

By given condition $\langle f'(\hat{x}), \frac{\hat{x}\,x}{\hat{x}-x} \rangle \leq 0$ and thus we must have $\langle f'(\hat{x}), \frac{xy}{y-x} \rangle \leq 0$.

This inequality is not compatible with (3.4).

Therefore $L = \phi$.

Hence the proof.

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