On the Calculus of Dirac Delta Function with Some Applications

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Abstract : *In this paper, we present different properties of Dirac delta function, provided with simple proof and definite integral. we obtain some results on the derivative of discontinuous functions, provided with an important problem, to change the traditional mathematical approach to this. The concept of first-order differential equations is developing by using it, to obtain the solution, with some applications on real life problems.*

.**Keywords:** *Dirac delta function, generalized derivative, sifting problem, Laplace transform*. .

I. INTRODUCTION (SIZE 10 & BOLD)

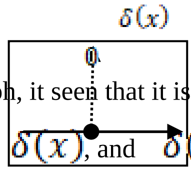
II. THE DIRAC DELTA FUNCTION WAS INTRODUCED BY P. DIRAC AT THE END OF 1920 S ,IN AN EFFORT TO CREATE MATHEMATICAL TOOL FOR DEVELOPING THE FIELD OF QUANTUM THEORY[1] .IT CAN BE REGARDED AS A GENERALIZED OR PROBABILITY FUNCTION, IN DYNAMICS IT KNOWN AS IMPULSE FUNCTION [2].IT IS A VERY USEFUL MATHEMATICAL TOOL THAT APPEAR IN MANY PLACES IN PHYSICS SUCH AS QUANTUM MECHANICS, ELECTROMAGNETISM, OPTICS, ENGINEERING PROBLEMS[2] .VARIOUS WAY OF DEFINING DIRAC DELTA FUNCTION AND ITS APPLICATION TO THE DETERMINATION OF THE DERIVATIVE OF DISCONTINUOUS FUNCTION (SEE [3],[4]), FOR SOME INTERESTING APPLICATION DISCUSSION IN STATISTIC (SEE [1],[5]),IT PLAYS AN IMPORTANT ROLE IN THE IDEALIZATION OF AN IMPULSE IN RADAR CIRCUITS[6],IT CAN BE VIEWED AS A TOOL OF MAKING CALCULUS LOOK LIKE OPERATION[7].SOME RELATED PROPERTIES ARE PRESENTED WITH SOME APPLICATION TO THE DIFFRACTION OVER CURVES AND SURFACES[8]. QAYAM INTRODUCED SOME RIGOROUSLY PROVE FOR THE PURPOSE OF CALCULATING LAPLACE TRANSFORM[9],[10] , FOR SOLVING INITIAL VALUE PROBLEM ,A NEW TECHNIQUE OF INTEGRAL OF DISCONTINUOUS FUNCTIONS USING DIRAC DELTA FUNCTION AND FOURIER TRANSFORM [11].SALIENT PROPERTIES OF THIS FUNCTION ARE LISTED AND DISCUSSED [12].THIS PAPER TARGETS TO STUDY AN IMPORTANT PROPERTIES THAT RELATED TO CALCULUS, WITH SOME APPLICATION ON REAL LIFE PROBLEM ,IT DIFFERS FROM OTHERS SINCE IT DEALS WITH LARGE NUMBERS OF PROPERTIES PROVIDED BY SIMPLE PROOFS AND AN IMPORTANT USAGE.

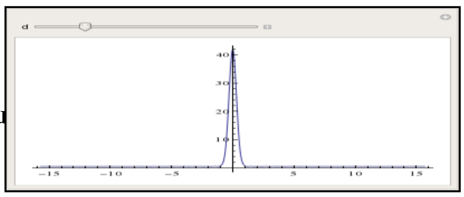
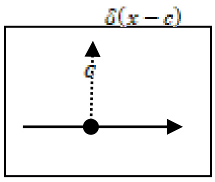
Definition1.1: (see [2], [3]) Dirac delta function can be defined as follows:

$\delta(x) = \begin{cases} 0 \ . if\ x \neq 0 \\ \infty \ . if\ x = 0 \end{cases}$, such that $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$,

And for impulse at $x = c$, we have:

$\delta(x-c) = \begin{cases} 0 \ . if\ x \neq c \\ \infty \ . if\ x = c \end{cases}$, such that $\int_{-\infty}^{\infty} \delta(x-c)\, dx = 1$, and its graph as follows:



From the graph, it seen that it is an even fu

$\delta(-x) = \delta(x)$, and $\delta(x-a)$

The properties of Dirac delta function 1.2:

(1) $f(x) \cdot \delta(x) = f(0) \cdot \delta(x)$.

$f(x-a) \cdot \delta(x-a) = f(a) \cdot \delta(x-a)$.

(2) Sifting property: $\int_A^B f(x)\delta(x)\, dx = \begin{cases} f(0) \ .A < 0 < b \\ \ 0 \quad . \ 0 \notin (A, B) \end{cases}$

Proof :(see [13]).

By using integration by parts, let $A < 0 < b$

$$\int_A^B f(x)\delta(x)\,dx = H(x)f(x)\Big|_A^B - \int_A^B H(x)f'(x)\,dx$$

$$= H(B)f(B) - H(A)f(A) - \int_A^B H(x)f'(x)\,dx$$

$$= H(B)f(B) - \int_A^B H(x)f'(x)\,dx \;(H(x) = 0.\, when\ x < 0)$$

$$= f(B) - \int_A^B H(x)f'(x)\,dx \;,(H(x) = 1, when\ x > 0)$$

$$= f(B) - \int_0^B f'(x)\,dx \;,(H(x) = 1, when\ x > 0)$$

$$= f(B) - f(x)\Big|_0^B \;,\ Fundamental\ theorem\ of\ calculus)$$

$$= f(B) - \big(f(B) - f(0)\big)$$

$$= f(B) - f(B) + f(0)$$

$$= f(0)$$

Another proof:

$$\int_A^B f(x)\delta(x)\,dx = \int_A^B f(0)\delta(x)\,dx \text{, (product properties of } \delta(x)\text{)}$$

$$= f(0)\int_A^B \delta(x)\,dx$$

$$= f(0)\cdot 1$$

$$= f(0)$$

(3) $$\int_A^B f(x)\delta(x-a)\,dx = \begin{cases} f(a) & .A < a < B \\ 0 & .\ a \notin (A.B) \end{cases}$$

Proof:

We have: $f(x-a)\cdot\delta(x-a) = f(a)\cdot\delta(x-a)$, a ∈ (A, B)

$$\int_A^B f(x)\delta(x-a)\,dx = \int_A^B f(a)\delta(x-a)\,dx \text{, a} \in \text{(A, B)}$$

$$= f(a)\int_A^B \delta(x-a)\,dx \text{, a} \in \text{(A, B)}$$

$$= f(a)\cdot 1$$

$$= f(a).$$

(4) $$\int_A^B f(x)\delta(kx)\,dx = \frac{1}{|k|}f(0)$$

Proof:

We have: $f(x)\cdot\delta(kx) = f(0)\cdot\delta(kx)$,

Assume that: $u = kx$

$du = k\,dx$

$$dx = \frac{du}{k}, \text{ then,}$$

$$\int_{-\infty}^{\infty} f(x)\delta(kx)\,dx = \int_{-\infty}^{\infty} f(0)\delta(kx)\,dx$$

$$= f(0)\int_{A}^{B} \delta(u) \cdot \frac{du}{k}$$

$$= \frac{1}{k} f(0)\int_{A}^{B} \delta(u) \cdot du$$

$$= \frac{1}{k} \cdot f(0)$$

(5) Composite property: (see [12]): $\delta\big(f(x)\big) = \sum_{i=1}^{n} \frac{\delta(x-x_i)}{|f'(x)|}$.

Proof:

Let, $f(x) = y$

$$X = f^{-1}(y).$$

$$dx = \frac{dy}{f^{-1}(x)}.$$

$$\int_{-\infty}^{\infty} g(x) \cdot \delta\big(f(x)\big)dx = \int_{-\infty}^{\infty} g(f^{-1}(y)) \cdot \delta(y) \frac{dy}{f'(f^{-1}(y))}$$

In this integral wherever y occur, is should be set zero.

$$\int_{-\infty}^{\infty} g(x) \cdot \delta\big(f(x)\big)dx = \frac{g(f^{-1}(0))}{f'(f^{-1}(0))}$$

As $f^{-1}(0) = x \Rightarrow f(x_i) = 0$, where x_i are roots.

$$\int_{-\infty}^{\infty} g(x) \cdot \delta\big(f(x)\big)dx = \sum_{i=1}^{n} \frac{g(x_i)}{f'(x_i)}, \text{ then}$$

$$\int_{-\infty}^{\infty} g(x) \cdot \delta\big(f(x)\big)dx = \int_{-\infty}^{\infty} \frac{\sum_{i=1}^{n} g(x) \cdot \delta(x-x_i)}{f'(x_i)}$$

$$= \int_{-\infty}^{\infty} g(x) \frac{\sum_{i=1}^{n} \delta(x-x_i)}{f'(x_i)}$$

By comparing both sides, we get:

$$\delta\big(f(x)\big) = \sum_{i=1}^{n} \frac{\delta(x-x_i)}{f'(x_i)}.$$

$$= \frac{\delta(x-x_1)}{|f'(x_1)|} + \frac{\delta(x-x_2)}{|f'(x_2)|} + \frac{\delta(x-x_3)}{|f'(x_3)|} + \ldots + \frac{\delta(x-x_n)}{|f'(x_n)|}.$$

We note that, all the following propositions are special case of this property.

(6) **Scaling property:** (see[6]) $\boldsymbol{\delta(k \cdot x) = \frac{1}{|k|}\delta(x)}$

Proof:

Let, $u = kx$

$$du = d(kx) = kd(x)$$

$$dx = \frac{d(u)}{k}, k \neq 0 . \Rightarrow |k| = \mp k..$$

$$\int_{-\infty}^{\infty} \delta(kx)\, dx = \int_{-\infty}^{\infty} \delta(u) \cdot \frac{d(u)}{k}$$

$$= \frac{1}{|k|} \int_{-\infty}^{\infty} \delta(u) \cdot d(u)$$

$$= \frac{1}{|k|} \int_{-\infty}^{\infty} \delta(x) \cdot d(x)$$

$$= \int_{-\infty}^{\infty} \frac{1}{|k|} \delta(x) \cdot d(x),$$

then by comparing both sides, we get: $\delta(k \cdot x) = \frac{1}{|k|} \delta(x)$.

Corollary 1.3: For $f(x) = kx$, which has a unique zero $= 0$,and such that $f'(0) = k$, then by using property (4), we get the following:

$$\delta(f(x)) = \frac{\delta(x-0)}{f'(x_1)}$$

$$\delta(k(x)) = \frac{\delta(x-x_1)}{f'(0)}$$

$$= \frac{\delta(x)}{|k|}$$

$$\therefore \ \delta(k(x)) = \frac{1}{|k|} \delta(x).$$

(7) $\delta(x^2 - a^2) = \frac{1}{2|a|} \big(\delta(x-a) + \delta(x+a)\big)$.[5]

Proof:

Let, $x = \sqrt{u}$, then $dx = \frac{1}{2\sqrt{u}} du$, and so:

$$\int_0^{\infty} f(x) \cdot \delta(x^2 - a^2)\, dx = \int_0^{\infty} f(\sqrt{u}) \cdot \delta(u - a^2) \frac{1}{2\sqrt{u}} du$$

$$= \frac{1}{2\sqrt{u}} \cdot f(|a|)$$

$$= \frac{f(|a|}{2\sqrt{u}}. \qquad (1)$$

Similarly, let $x = -\sqrt{u}$, then $dx = \frac{-1}{2\sqrt{u}} du$

$$\int_{-\infty}^{0} f(x) \cdot \delta(x^2 - a^2)\, dx = \int_0^{\infty} f(-\sqrt{u}) \cdot \delta(u - a^2) \frac{-1}{2\sqrt{u}} du$$

$$= f(-|a|) \cdot \frac{-1}{2\sqrt{u}}$$

$$= \frac{-f(-|a|)}{2\sqrt{u}} \qquad (2)$$

From (1) and (2), we have:

$$\int_{-\infty}^{\infty} f(x)\cdot\delta(x^2-a^2)\,dx =$$

$$= \int_0^{\infty} f(x)\cdot\delta(x^2-a^2)\,dx + \int_{-\infty}^{0} f(x)\cdot\delta(x^2-a^2)\,dx$$

$$= \frac{f(|a|}{2\sqrt{u}} + \frac{-f(-|a|)}{2\sqrt{u}}$$

$$= \frac{1}{2|a|}\big(\delta(a)+\delta(-|a|)\big)$$

$$= \int_{-\infty}^{\infty} \frac{1}{2|a|}\big(\delta(x-a)+\delta(x+a)\big)f(x)d(x).$$

$$\therefore \delta(x^2-a^2) = \frac{1}{2|a|}\big(\delta(x-a)+\delta(x+a)\big).$$

(8) Scaling property: $\delta\big(k(x-x_0)\big) = \frac{1}{|k|}\delta(x-x_0)$.

Proof:

Let: $y = k(x-x_0).c > 0$

$$y = kx - kx_0$$

$$x = \frac{y}{k} + x_0$$

$$dx = \frac{1}{k}dy$$

$$\int_{-\infty}^{\infty} f(x)\,\delta\big(k(x-x_0)\big)dx = \int_0^{\infty} f\left(\frac{y}{k}+x_0\right)\cdot\delta(y)\cdot\frac{1}{k}dx$$

$$= \frac{1}{k}\int_0^{\infty} f\left(\frac{y}{k}+x_0\right)\cdot\delta(y)\cdot dx$$

$$= \frac{1}{k}\int_0^{\infty} f(x_0).$$

Assume that: $c < 0$,

$$\int_{-\infty}^{\infty} f(x)\,\delta\big(k(x-x_0)\big)dx = \int_{-\infty}^{0} f\left(\frac{y}{k}+x_0\right)\cdot\delta(y)\cdot\frac{1}{k}dx$$

$$= \frac{1}{k}\int_0^{\infty} f(-x_0)$$

$$\int_{-\infty}^{\infty} f(x)\,\delta\big(k(x-x_0)\big)dx = \int_{-\infty}^{\infty} f(x)\cdot\frac{1}{|k|}\delta\big((x-x_0)\big)dx.$$

$$\therefore \delta\big(k(x-x_0)\big) = \frac{1}{|k|}\delta(x-x_0).$$

(9) $\delta((kx\pm b)) = \frac{1}{|k|}\delta\left(x\pm\frac{b}{k}\right)$.

Proof:

Let: $y = kx \pm b.\quad k \neq 0$

$$x = \frac{y\pm b}{k}$$

$$dy = kdy$$

$$\int_{-\infty}^{\infty} f(x)\,\delta\big((kx \pm b)\big)dx = \int_{-\infty}^{\infty} f\left(\frac{y \pm b}{k}\right)\cdot \delta(y)\cdot \frac{1}{k}dy$$

$$= \frac{1}{k}\cdot f\left(\frac{\pm b}{k}\right)$$

$$= \frac{1}{|k|}\cdot \int_{-\infty}^{\infty} f(x)\cdot \delta\left(x \pm \frac{b}{k}\right)\cdot dx$$

$$= \int_{-\infty}^{\infty} f(x)\cdot \frac{1}{|k|}\cdot \delta\left(x \pm \frac{b}{k}\right)\cdot dx$$

Then, by comparing both sides, we get

$$\therefore\ \delta((kx \pm b)) = \frac{1}{|k|}\delta\left(x \pm \frac{b}{k}\right).$$

(10) $$\delta(kx^2 - b) = \frac{1}{2k\sqrt{\frac{b}{a}}}\left[\delta\left(x + \sqrt{\frac{b}{a}}\right) + \delta\left(x - \sqrt{\frac{b}{a}}\right)\right].$$

Proof:

Let, $y = kx^2 - b. \quad k \neq 0$

$$x = \pm\sqrt{\frac{y+b}{k}}$$

$$dy = 2kdx$$

Then, we have

$$\int_{-\infty}^{\infty} f(x)\,\delta(kx^2 - b)dx \ = \int_{-\infty}^{\infty} f\left(\pm\sqrt{\frac{y+b}{k}}\right)\cdot \delta(y)\cdot \frac{1}{2k}dy$$

$$= \frac{1}{2|k|}\cdot f\left(\frac{\pm b}{k}\right).$$

$$= \frac{1}{2|k|}\cdot \int_{-\infty}^{\infty} f(x)\cdot \delta\left(x - \sqrt{\frac{b}{k}}\right) + \delta\left(x + \sqrt{\frac{b}{k}}\right)\cdot dx$$

$$= \int_{-\infty}^{\infty} f(x)\cdot \frac{1}{2|k|}\cdot \delta\left(x + \sqrt{\frac{b}{k}}\right) + \delta\left(x - \sqrt{\frac{b}{k}}\right)\cdot dx$$

Then, by comparing both sides, we get:

$$= \frac{1}{2|k|}\cdot \delta\left(x + \sqrt{\frac{b}{k}}\right) + \delta\left(x - \sqrt{\frac{b}{k}}\right)\cdot$$

$$\therefore\ \delta(kx^2 - b) = \frac{1}{2k\sqrt{\frac{b}{a}}}\left[\delta\left(x + \sqrt{\frac{b}{a}}\right) + \delta\left(x - \sqrt{\frac{b}{a}}\right)\right].$$

(11) $\delta\big((x-a)(x-b)\big) = \frac{1}{|a-b|}[\delta(x+a)+\delta(x-b)]$

Proof:

Let, $y = (x-a)(x-b)$

$$y = x^2-(a+b)x+ab$$

$$dy = [2x-(a+b)]dx$$

$$dx = \frac{dy}{2x-(a+b)},$$

Then we have, $x^2-(a+b)x+(ab-y)=0$

$$x = \frac{(a+b)\pm\sqrt{(a+b)^2-4(ab-y)}}{2}$$

$$dx = \frac{dy}{(a+b)\pm\sqrt{(a+b)^2-4(ab-y)}-(a+b)}$$

Then, we have:

$$\int_{-\infty}^{\infty} f(x)\,\delta\big((x-a)(x-b)\big)dx = \int_{-\infty}^{\infty} f\left(\frac{(a+b)\pm\sqrt{(a+b)^2-4(ab-y)}}{2}\right)\cdot\delta(y)\cdot\frac{1}{2k}\cdot\frac{dy}{(a+b)\pm\sqrt{(a+b)^2-4(ab-y)}-(a+b)}$$

$$= f\left(\frac{(a+b)\pm\sqrt{(a+b)^2-4(ab-y)}}{2}\right)\cdot\frac{1}{(a+b)\pm\sqrt{(a+b)^2-4(ab-y)}-(a+b)}$$

$$= f\left(\frac{(a+b)\pm(a-b}{2}\right)\cdot\frac{1}{|a-b|}$$

$$= \big(f(a)+f(b)\big)\cdot\frac{1}{|a-b|}$$

$$= \int_{-\infty}^{\infty} f(x)\big(\delta(x-a)+\delta(x-b)\big)\cdot\frac{1}{|a-b|}dx$$

$$\therefore \delta\big((x-a)(x-b)\big) = \frac{1}{|a-b|}[\delta(x+a)+\delta(x-b)]$$

Problem 1.4:

show that: $\int_{-1}^{1} x^2\delta(3x+1)\,dx = \frac{1}{27}.$

Proof:

Let, $y = 3x+1, \Rightarrow x = \frac{y-1}{3}$, then $dy = 3dx$

$$\int_{-1}^{1} x^2\delta(3x+1)\,dx = \int_{-2}^{4}\left(\frac{y-1}{3}\right)^2\delta(y)\frac{dy}{3}$$

$$= \frac{1}{3}\int_{-2}^{4}\left(\frac{y-1}{3}\right)^2\delta(y)\,dy = \frac{1}{3}\cdot\frac{1}{9} = \frac{1}{27}$$

1. Generalized function:

We have $H(t) = \begin{cases} 0 \,.\, t < 1 \\ 1 \,.\, t > 1 \end{cases}$, then given that, $H(t) = \int_{-\infty}^{t} \delta(t)dt$, by using the fundamental theorem of calculus: We have, $H'(t) = \delta(t)$. This means that a jump discontinuity contributes delta function to generalized derivative. Then to each jump discontinuity, adds a delta function scaled by the size of the jump to $f'(t)$ .therefore the derivative of discontinuous function is defined as a sum of regular function and linear combination of delta function (singular point) as follows:

Theorem 2.1: (see [3]) For $f(x)$ is a continuous everywhere except for $x = a$, and, $\lim_{x\to a^+} f(x) - \lim_{x\to a^-} f(x) = k$, where k is the value of the gap at $x = a$, then $f'(x) = \emptyset(x) + k \cdot \delta(x-a)$, where $\emptyset(x)$ is the derivative of $f(x)$ on its domain, except at $x = a$.

Problem 2.2:

For $f(x) = \begin{cases} x^2 + 3 . x < 0 \\ e^{-x} . x > 0 \end{cases}$, prove that: $f'(x) = -2\delta(x) + \begin{cases} 2x + 3 . x < 0 \\ -e^{-x} . x > 0 \end{cases}$

Proof:

The gap occurs at $= 0$, then, the value of the gap at $(x = 0)$

$$= \lim_{x\to 0^+} f(x) - \lim_{x\to 0^-} f(x) = 1 - 3 = -2,$$

Therefore, $k = -2$,from the theorem, then we have:

$$f'(x) = \emptyset(x) + k\delta(x-a), \emptyset(x) = \begin{cases} 2x + 3 . x < 0 \\ -e^{-x} . x > 0 \end{cases}$$

Therefore, $f'(x) = -2\delta(x) + \begin{cases} 2x + 3 . x < 0 \\ -e^{-x} . x > 0 \end{cases}$.

Problem 2.3:

Using the graph of $f(x)$ besides,

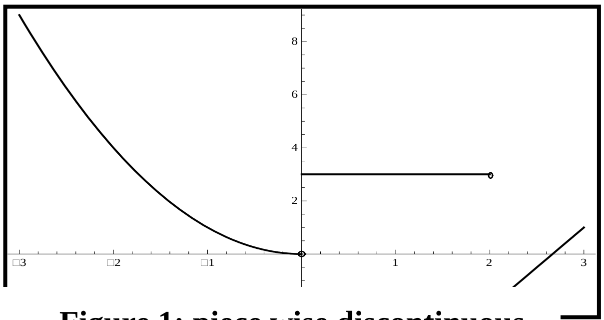


Figure 1: piece wise discontinuous function

prove that:

$$f'(x) = 3\delta(x) + 5\delta(x-2) + \begin{cases} 2x. \ \ x < 0 \\ 0.0 < x < 2 \\ 3. \ \ x > 2 \end{cases}$$

Proof:

$$f(x)=\begin{cases}x^2\ .if\quad x<0\\ 3.\ if\ \ 0<x<2\\ 3x-8\,.if\ x>2\end{cases}$$

The gap occurs at $(x=0)$

the value of the gap at $(x=0)$

$$=\lim_{x\to 0^+} f(x)-\lim_{x\to 0^-} f(x)=3-0=3,$$

Also, the gap occurs at $(x=2)$

the value of the gap at $(x=2)$

$$=\lim_{x\to 2^+} f(x)-\lim_{x\to 2^-} f(x)=-2-3=-5,$$

from the theorem, we have:

$$f'(x)=\emptyset(x)+k_1\delta(x-a_1)++k_2\delta(x-a_2)$$

Then we have, $k_1\ =\ 3$, $k_2\ =\ 3$, $a_1=0.\ \ a_2=-5$

Therefore, $f'(x)=3\delta(x)+-5\delta(x-2)+\begin{cases}2x+3.\,x<0\\ -e^{-x}\ .\,x>0\end{cases}$

Corollary 2.4: if $f(x)$ is a continuous piece wise everywhere except for $x=a_i\ where,$ $i=1.2.3.\ldots.n$, then there exists a delta function, such that:

$$f'(x)=\ \emptyset(x)+\sum_{i=1}^{i=n} k_i\cdot\delta(x-\mathrm{a}_i)$$

where, k_i are the values of the gaps at $x=\mathrm{a}_i$, such that:

$k_i=\lim_{x\to \mathrm{a}_i{}^+} f(x)-\lim_{x\to \mathrm{a}_i} f(x)$), and $\emptyset(x)$ is the derivative of $f(x)$ on its domain, except at $x=a_i$.

theorem 2.5: see [12]: let $(x)\ =|x|$, then $\delta(x)=\frac{1}{2}\frac{d^2x}{dx^2}$.

Proof:

$$f(x)=\begin{cases}x.\quad x>0\\ -x.\quad x<0\end{cases},\quad \Rightarrow f'(x)=\begin{cases}1.x>0\\ -1\,.x<0\end{cases},$$

$$f''(x)=\ 0+2\delta(x)$$

$$f''(x)=\ 2\delta(x)\ .$$

$$\delta(x)=\frac{f''(x)}{2}$$

$$\therefore\ \delta(x)=\frac{1}{2}\frac{d^2(f(x))}{dx^2}$$

$$\delta(x)=\frac{1}{2}\frac{d^2(|x|)}{dx^2}$$

Corollary 2.6:

For $f(x) = |x-a|$, $a \in R$, then $\delta(x-a) = \frac{1}{2}\frac{d^2(|x-a|)}{dx^2}$.

Proof:

$$nf(x) = \begin{cases} x-a. & x>a \\ a-x. & x<a \end{cases}, \Rightarrow f'(x) = \begin{cases} 1.x>a \\ -1.x<a \end{cases},$$

$$f''(x) = 0 + 2\delta(x)$$

$$f''(x) = 2\delta(x-a)$$

$$\delta(x-a) = \frac{f''(x-a)}{2}$$

$$\therefore \delta(x-a) = \frac{1}{2}\frac{d^2(f(x))}{dx^2}$$

$$\therefore \delta(x-a) = \frac{1}{2}\frac{d^2(|x-a|)}{dx^2}.$$

Corollary 2.7: For $f(x) = |x-a_1| + |x-a_2|$, $a_1.a_2 \in R$, then

$$\delta(x-a_1) + \delta(x-a_2) = \frac{1}{2}\frac{d^2(|x-a_1|+|x-a_2|)}{dx^2}.$$

Proof:

$$f(x) = \begin{cases} -2x+(a_1+a_2).if\ x<a_1 \\ a_2-a_1.if\ a_1<x<a_2 \\ 2x-(a_1+a_2).if\ x>a_2 \end{cases}, \Rightarrow f'(x) = \begin{cases} -2.if\ x<a_1 \\ 0.if\ a_1<x<a_2 \\ 2.if\ x>a_2 \end{cases}$$

$$f''(x) = 0 + 2\delta(x-a_1) + 2\delta(x-a_2)$$

$$f''(x) = 2[\delta(x-a_1) + \delta(x-a_2)]$$

$$\delta(x-a_1) + \delta(x-a_2) = \frac{f''(x)}{2}$$

$$\therefore \delta(x-a_1) + \delta(x-a_2) = \frac{1}{2}\frac{d^2(|x-a_1|+|x-a_2|)}{dx^2}.$$

Corollary 2.8: For $f(x) = \sum_{i=1}^{n}|x-a_i|$, $a_i \in .R$, then

$$\delta(x-a_i) = \frac{1}{2}\frac{d^2\sum_{i=1}^{n}(|x-a_i|)}{dx^2}.$$

2. The Laplace transform of delta function:

Since the Laplace transform is given by the integral, it should be easy to compute it for the delta function as follows:

i. $\mathcal{L}(\delta(t)) = 1$

ii. $\mathcal{L}(\delta(t-a)) = e^{-as}, a>0$

Solution:

i. $$\mathcal{L}\big(\delta(t)\big) = \int_{0^-}^{\infty} \delta(t)\, e^{-st}\, dt = e^{-s(0)} = e^{0} = 1$$

ii. $$\mathcal{L}\big(\delta(t-a)\big) = \int_{0^-}^{\infty} \delta(t-a)\, e^{-st}\, dt = e^{-s(a)} = e^{-sa}.$$

The two formulas are consistent, if we set $= 0$, in formula (2), the we get formula (1).

Theorem 3.1: see [10]. Let, $f(t)\colon R \to R$, be a function, let $\delta(t-a)$ be the Dirac delta function, let $c \geq 0$, then the Laplace transform of their product is:

$$\mathcal{L}(\delta(t-c)\cdot f(t)(s)) = e^{-sc}\cdot f(c)$$

Proof:

$$\mathcal{L}(\delta(t-c)\cdot f(t)(s)) = \int_0^{\infty} e^{-st}\cdot \delta(t-c) f(t) dt.$$

$$= \int_0^{\infty} e^{-sc}\cdot \delta(t-c) f(c) dt.$$

$$= e^{-sc}\cdot f(c) \int_0^{\infty} \delta(t-c)\, dt$$

$$= e^{-sc}\cdot f(c)\cdot 1$$

$$= e^{-sc}\cdot f(c)$$

Corollary 3.2: For the differential equation: $\bar{y} + ay = \sum_{i=1}^{n} f(t_i)\,\delta(t-t_i)$, where $y(t) = 0$,for $t < 0$, then the solution is:

$$y(t) = y(0)e^{-at} = \sum_{i=1}^{n} f(t_i)\cdot e^{-a(t-t_i)}\cdot u(t-t_i)$$

Proof:

By using Laplace to both sides, we get:

$$s\mathcal{L}(y) - y(0) + a\mathcal{L}(y) = \sum_{i=1}^{n} f(t_i)\cdot e^{-st_i}$$

$$\mathcal{L}(y)(s+a) = y(0) + \sum_{i=1}^{n} f(t_i)\cdot e^{-st_i}$$

$$\mathcal{L}(y) = \frac{y(0)}{s+a} + \frac{\sum_{i=1}^{n} f(t_i)\cdot e^{-st_i}}{s+a}$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{y(0)}{s+a}\right) + \mathcal{L}^{-1}\left(\frac{\sum_{i=1}^{n} f(t_i)\cdot e^{-st_i}}{s+a}\right)$$

$$\therefore y(t) = y(0)e^{-at} = \sum_{i=1}^{n} f(t_i)\cdot e^{-a(t-t_i)}\cdot u(t-t_i).$$

Problem 3.3:

Some cattle of 1600 sheep with 8% growth of rate, it has been decided to get rid of this cattle in two years' time, by selling the same numbers every 6 months starting from now. How many should be sold at time t ?.

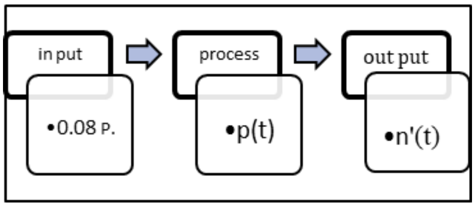
Proof:

Let, $P(t) \equiv$ the number of cattle at time t .

The growth rate (input) = 0.08 P.

Let, $N \equiv$ number sold each 6 month.

The total number sold at a time t is:



$$n(t) = N \cdot u\left(t - \tfrac{1}{2}\right) + N \cdot u(t-1) + N \cdot u\left(t - 1\tfrac{1}{2}\right) + N \cdot u(t-2)$$

The selling rate (output) is:

$$n'(t) = N \cdot \delta\left(t - \tfrac{1}{2}\right) + N \cdot \delta(t-1) + N \cdot \delta\left(t - 1\tfrac{1}{2}\right) + N \cdot \delta(t-2)$$

Then the differential equation that models this is :

$\frac{dp}{dt} =$ the rate of input – the rate of out put

$$\frac{dp}{dt} = 0.08\,p - N \cdot u\left(t - \tfrac{1}{2}\right) - N \cdot u(t-1) - N \cdot u\left(t - 1\tfrac{1}{2}\right) - N \cdot u(t-2)$$

Then our I.V.P. is:

$$\mathcal{L}\left(\frac{dp}{dt} - 0.08p\right) = \mathcal{L}\left(-N\delta(t-0.5) - N\delta(t-1) - N\delta(t-1.5) - N\delta(t-2)\right)$$

$$P(0) = 1600.$$

By using Laplace transform to both sides, we get:

$$s\mathcal{L}(p) - p(0) - 0.08\mathcal{L}9p) = -N\left(e^{-\frac{1}{2}s} + e^{-s} + e^{-1.5s} + e^{-2s}\right).$$

$$\mathcal{L}(p)(s - 0.08) = 1600 - N\left(e^{-\frac{1}{2}s} + e^{-s} + e^{-1.5s} + e^{-2s}\right)$$

$$\mathcal{L}(p) = \frac{1600}{s-0.08} - N\left(\frac{e^{-\frac{1}{2}s}}{s-0.08} + \frac{e^{-s}}{s-0.08} + \frac{e^{-1.5s}}{s-0.08} + \frac{e^{-2s}}{s-0.08}\right).$$

By taking the inverse of Laplace transform to both sides, we get:

$$p(t) = 1600e^{0.08t} - \mathrm{N}\left(e^{.0.08(t-\frac{1}{2})} \cdot u\left(t - \tfrac{1}{2}\right) + e^{.0.08(t-1)} \cdot u(t-1) + e^{.0.08(t-1.5)} \cdot u(t-1.5) + e^{.0.08(t-2)} \cdot u(t-2)\right)$$

To find N, we use the fact that: $p(2^+) = 0$

$$0 = 1600e^{0.16} - \mathrm{N}(e^{.0.12} \cdot + e^{.0.08} + e^{.0.04} + 1)$$

$$\mathrm{N} = \frac{1600 \cdot e^{0.16}}{e^{.0.12} \cdot + e^{.0.08} + e^{.0.04} + 1} \approx 442.$$

Therefore, the number sold of cattle each 6 months ≈ 442.

CONCLUSION: To sum up, several properties of Dirac delta function, were discussed, and the derivative of discontinuous functions were studied, confirmed by many examples. We obtained a method to solve ordinary differential equations, with distinguished application on time rate problems.

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