(3,2) - Jection Operator and its Properties

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Abstract :

In this paper, we introduce a new type of linear operator called (3,2)-jection operator on a linear space and then we find an innovative result.

Key words :

Linear operator, Projection operator, (3,2)-jection operator.

Introduction :

It is well known that projection plays a pivatal role in functional analysis and linear algebra hereat to develop functional analysis and linear algebra, we take care of the tractable generalisation of this operator.

In this paper, we introduce a new type of linear operator called a (3, 2)-jection operator which is a generalisation of projection operator in the sense that every projection is a (3, 2)-jection operation but a (3, 2)-jection operator is not necessarily a projection. Here we study (3, 2)-jection on R^2 or C^2 and find an innovative result.

Requisite :

Linear operator : The operator T on a linear space L is said to be a linear operator if $T(\alpha x+\beta y)=\alpha T(x)+\beta T(y)$ for all $x, y \in L$ and for scalars α and β.

Projection operator : The operator T on a linear space L is a projection on some subspace M of L if $T^2=T$.

(3, 2)-jecion operator : The operator E on a linear space L is said to be a (3, 2)-jection operator if $E^3 = E^2$.

Theorem : Let E be a (3, 2)-jection on R^2 and C^2 then either E is a projection or $E^2=0$.

Proof : Let E (x, y) = (ax + by, cx + dy)

where $(x, y) \in R^2 \text{ or } C^2$

and a, b, c, d are scalars.

We have,

$$\begin{aligned} E^2(x,y) &= E(E(x,y)) \\ &= E(ax+by, cx+dy) \\ &= (a(ax+by)+b(cx+dy), c(ax+by)+d(cx+dy)) \\ &= ((a^2+bc)x+(ab+bd)y, (ac+cd)x+(bc+d^2)y) \end{aligned}$$

$$= \left(px + qy, rx + sy\right)$$

where,

$$p = a^2 + bc \ldots\ldots(1)$$

$$q = ab + bd \ldots\ldots(2)$$

$$r = ac + cd \ldots\ldots(3)$$

$$s = bc + d^2 \ldots\ldots(4)$$

and,

$$E^3(x, y) = E\left(E^2(x, y)\right)$$

$$= E\left(px + qy, rx + sy\right)$$

$$= \left(a\left(px + qy\right) + b\left(rx + sy\right), c\left(px + qy\right) + d\left(rx + sy\right)\right)$$

$$= \left(\left(ap + br\right)x + \left(aq + bs\right)y, \left(cp + dr\right)x + \left(cq + ds\right)y\right)$$

For E to be a (3, 2)-jection, we have

$E^3 = E^2$

Comparing the co-efficients of like terms, we get

$$ap + br = p \Rightarrow \left(a - 1\right)p + br = 0 \ldots\ldots(5)$$

$$\& \ aq + bs = q \Rightarrow \left(a - 1\right)q + bs = 0 \ldots\ldots(6)$$

$$\& \ cp + dr = r \Rightarrow cp + \left(d - 1\right)r = 0 \ldots\ldots(7)$$

$$\& \ cq + ds = s \Rightarrow cq + \left(d - 1\right)s = 0 \ldots\ldots(8)$$

Here we consider the following two cases:

Case-1 : When a = 1

Then from (5) and (6), we get

$br = 0$ and $bs = 0$

Therefore, either $b = 0$ or $r = s = 0$.

Case-1.1. : By taking b = 0, we get

$p = 1$...{from (1)}

$q = 0$ {from (2)}

$r = c\,(1 + d)$ {putting a = 1 in (3)}

$s = d^2$ {from (4)}

Putting $p = 1$ and $r = c\,(1 + d)$ in (7), we get

$$c + (d - 1)\, c\, (1 + d) = 0$$

$$\Rightarrow c\left\{1 + \left(d - 1\right)\left(d + 1\right)\right\} = 0$$

$$\Rightarrow c\left\{1 + d^2 - 1\right\} = 0$$

$$\Rightarrow cd^2 = 0$$

Thus, either c = 0 or d = 0

When taking c = 0, we get

$(d-1)\, d^2 = 0$ {putting c = 0 and s = d^2 in (8)}

Either d = 0 or d = 1.

By observing all the above, we get the following sub-cases:

Subcase (1.1.1) : a = 1, b = 0, c = 0, d = 0

we find

$$E\,(x, y) = (x, 0)$$

and $E^2\,(x, y) = E\,(x, 0) = (x, 0)$

Here, $E^2 = E$

Hence E is projection

Sub-case (1.1.2) : $a = 1, b = 0, c = 0, d = 1$

So, we find

$E\,(x, y) = (x, y)$ which is obviously a projection.

Sub-case (1.1.3) : $a = 1, b = 0, c \neq 0, d = 0$

Here, we find

$$E\,(x, y) = (x, cx)$$

and $E^2\,(x, y) = E\,(x, cx)$

$$= (x, cx)$$

Hence, $E^2 = E$

This means E is a projection

Case 1.2 : When r = s = 0 i.e. $b \neq 0$

From (3) and (4), we get

$c\,(1 + d) = 0$ and $bc + d^2 = 0$

$\Rightarrow (c = 0 \text{ or } d = -1)$ and $bc + d^2 = 0$

By taking c = 0 and $bc + d^2 = 0$, we get

$$d = 0$$

and by taking d = –1 and $bc + d^2 = 0$, we get

$$bc = -1$$

$$\Rightarrow c = \frac{-1}{b}$$

Thus, we get the following sub-cases:

Sub-case 1.2.1 : a = 1, c = 0, d = 0

then we find

$$E\,(x, y) = (x + by, 0)$$

and $E^2(x, y) = E(x + by, 0)$

$$= (x + by, 0)$$

Here $E^2 = E \Rightarrow E$ is a projection.

Sub-case 1.2.2 : When $a = 1, c \neq 0, b \neq 0, d = -1$

Here, we find

$$E(x, y) = \left(x + by, -\frac{1}{b}x - y\right)$$

and $E^2(x, y) = E\left(x + by, -\frac{1}{b}x - y\right)$

$$= \left(x + by - x - by, \frac{-x}{b} - y + \frac{x}{b} + y\right)$$

= (0, 0)

i.e. $E^2 = 0$

Hence, we conclude that when a = 1 then

Either E is a projection or $E^2 = 0$

Similarly, we can deal with the case when d = 1 because equations are symmetrical in a and d.

Case 2 : When $a \neq 1$

Now from (5), we get

(a–1)p = –br

$$\Rightarrow \frac{p}{r} = \frac{-b}{a-1} \quad \text{.......(2.1)}$$

and, from (7), we get

cp = –(d–1)r

$$\Rightarrow \frac{p}{r} = \frac{-(d-1)}{c} \quad \text{.......(2.2)}$$

from (2.1) and (2.2), we get

$$\frac{-b}{a-1} = \frac{-(d-1)}{c}$$

$$\Rightarrow bc = (a-1)(d-1)$$

$$\Rightarrow d - 1 = \frac{bc}{a-1}$$

$$\Rightarrow d = \frac{bc}{a-1} + 1$$

$$\Rightarrow d = \frac{bc + a - 1}{a-1} \quad \text{.......(2.3)}$$

Now, from (2), we get

$$q = b(a+d)$$

$$= b\left\{a + \frac{bc + a - 1}{a - 1}\right\} \qquad \{\text{using } (2.3)\}$$

$$= \frac{b}{a-1}\left\{a(a-1) + bc + a - 1\right\}$$

$$= \frac{b}{a-1}\left\{a^2 - \not{a} + bc + \not{a} - 1\right\}$$

$$= \frac{b}{a-1}\left\{a^2 + bc - 1\right\}$$

$$= \frac{b}{a-1}(p-1) \qquad \{\text{using } (1)\}\text{..........................}(2.4)$$

Again, from (3), we get

$$r = c(a+d)$$

$$= \frac{c}{a-1}(p-1) \text{ ....}(2.5)$$

Substituting the value of r from (2.5) in (5), we get

$$(a-1)p + b\left\{\frac{c}{a-1}(p-1)\right\} = 0$$

$$\Rightarrow (a-1)^2 p + bc(p-1) = 0$$

$$\Rightarrow (a-1)^2 p + (p - a^2)(p-1) = 0 \qquad \{\text{putting } bc = p - a^2 \text{ from } (1)\}$$

$$\Rightarrow p(a^2 - 2a + 1) + p^2 - p - a^2p + a^2 = 0$$

$$\Rightarrow pa^2 - 2ap + p + p^2 - p - a^2p + a^2 = 0$$

$$\Rightarrow p^2 - 2ap + a^2 = 0$$

$$\Rightarrow (p-a)^2 = 0$$

$$\Rightarrow p = a\text{..}(2.6)$$

Putting the value of p = a from (2.6) in (2.4), we get,

$$q = \frac{b}{a-1}(a-1)$$

i.e. $q = b$.. (2.7)

and, putting the value of p = a from (2.6) in (2.5), we get,

$$r = \frac{c}{a-1}(a-1)$$

i.e. $r = c$.. (2.8)

Now, substituting $bc = p - a^2$ from (1) in (2.3), we get

$$d = \frac{p - a^2 + a - 1}{a - 1}$$

$$= \frac{a - a^2 + a - 1}{a - 1} \qquad \{\text{putting } p = a \text{ from } (2.6)\}$$

$$= \frac{-a^2 + 2a - 1}{a - 1}$$

$$= \frac{-(a^2 - 2a + 1)}{a - 1}$$

$$= \frac{-(a - 1)^2}{a - 1}$$

$= -(a - 1)$

$= 1 - a$

i.e $d = 1 - a$.. (2.9)

Putting $d = 1 - a$ from (2.9) and $q = b$ from (2.7) in (6), we get

$$s = \frac{(1 - a)q}{b} = \frac{db}{b} = d$$

i.e. $s = d$..(2.10)

Finally, we get

$p = a,\ q = b,\ r = c,\ s = d$

and then

$$E^2(x, y) = (px + qy, rx + sy)$$

$$= (ax + by, cx + dy)$$

$= E\ (x, y)$

i.e. $E^2 = E$

$\Rightarrow$ E is a projection.

Furthermore, we observe that equation (5), (6) (7) and (8) are satisfied by

$p = q = r = s = 0$

In this case, we get

$$E^2(x, y) = (0, 0)$$

i.e. $E^2 = 0$

So, we need to show that this case exists also,

By setting $p = 0$, we get

$a^2 + bc = 0$ {from (1)}..(3.1)

By setting $s = o$, we get

$d^2 + bc = 0$ {from (4)} ....(3.2)

Subtracting (3.2) from (3.1), we get

$$a^2 - d^2 = 0$$

$$\Rightarrow (a - d)(a + d) = 0$$

$$\Rightarrow a = \pm d$$

Considering a = d, we get

$q = b(a + a)$ {putting d = a in (2)}

or $q = 2ab$

$\Rightarrow 0 = 2ab$ {providing q = 0}

$\Rightarrow$ Either a = 0 or b = 0

Taking a = 0, we get

d = 0 {as a = d}

From (3.1), we get

bc = 0

$\Rightarrow$ Either b = 0 or c = 0

Thus, we consider the following cases :

Case [3.1.1] :

Taking $a = 0, b \neq 0, c = 0, d = 0$

In this case, we get

$$E(x, y) = (by, 0)$$

and $E^2(x, y) = E(by, 0)$

$= (0, 0)$

i.e. $E^2 = 0$

Case [3.1.2] :

Taking $a = 0, b = 0, c \neq 0, d = 0$

In this case, we get

$$E(x, y) = (o, cx)$$

and $E^2(x, y) = E(0, cx)$

$= (0, 0)$

i.e $E^2 = 0$

Case [3.1.3] :

Taking $a = 0, b = 0, c = 0, d = 0$

In this case, we get

$$E(x, y) = (0, 0)$$

and $E^2(x, y) = (0, 0)$

i.e. $E^2 = 0$

Case [3.2.1] :

Taking $a \neq 0$ and $b = 0$

Then from equation (3.1), we get

$$a^2 = 0$$

$$\Rightarrow a = 0 \text{ which contradicts our assumption that } a \neq 0$$

so we take $b \neq 0$

Therefore, from equation (3.1), we get

$$c = \frac{-a^2}{b}$$

and from equation (2), we get

$$q = b(a + d)$$

$$0 = b(a + d) \qquad \{\text{because } q = 0\}$$

$$\Rightarrow a + d = 0$$

$$\Rightarrow d = -a \text{ ....}(3.3)$$

In this case, we observe also that

$r = c\,(a + d)$ {from (3)}

$= c\,(a - a)$ {from (3.3)}

$= 0$

and $s = bc + d^2$ {from (4)}

$= bc + a^2$ {from (3.3)}

$= 0$ {from (3.1)}

Thus from the above observation, we get

$$c = \frac{-a^2}{b} \text{ and } d = -a$$

Hence, $E(x, y) = \left(ax + by, \frac{-a^2}{b}x - ay \right)$

and $E^2(x, y) = E\left(ax + by, \frac{-a^2}{b}x - ay \right)$

$$= \left(a(ax + by) + b\left(\frac{-a^2}{b}x - ay \right), \frac{-a^2}{b}(ax + by) - a\left(\frac{-a^2}{b}x - ay \right) \right)$$

$$= \left((a^2 - a^2)x + (ab - ab)y, \left(\frac{-a^3}{b} + \frac{a^3}{b} \right)x + (a^2 - a^2)y \right)$$

$= (0, 0)$

i.e. $E^2 = 0$

Thus non-trivial cases also exist when $E^2 = 0$.

Hence, we conclude that a (3, 2)-jection E on R^2 or C^2 is either a projection or $E^2 = 0$.

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