

Solution of the Blasius Equation by using Adomain Mahgoub Transform

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Abstract— In this paper, we present solution of Blasius differential equation with condition at infinity and converted the series solution into rational function by using Padé's approximation. A new method is introduced, called Adomain Mahgoub Transform Method (ADMTM), which is a combination of Adomain Decomposition Method and Mahgoub Transform.

Keywords — Adomain Decomposition Mahgoub Transform Method (ADMTM), Blasius Equation, Padé's Approximation.

I. INTRODUCTION

THE RESULT OF BLASIU'S EQUATIONS TO FLUID FLOW IS VERY USEFUL FOR ENGINEERS, PHYSICISTS AND MATHEMATICIANS. IT IS THE BASIC EQUATION IN FLUID DYNAMICS [5]. IT IS THE MIXING LAYER THAT IS FOUND IN VISCOUS INCOMPRESSIBLE FLUID. THERE ARE TWO TYPES OF BLASIU'S EQUATIONS WHICH ARE SIMILAR TO DIFFERENTIAL EQUATION, BUT THE MAIN DIFFERENCE IS DIFFERENT BOUNDARY CONDITIONS [7]. THE MAIN DIFFICULTY IS IN CHANGING THE BOUNDARY CONDITIONS AT INFINITY. WE ARE USING PADÉ'S APPROXIMATION AT INFINITY [4]. PADÉ'S APPROXIMATION GIVES SOLUTION IN SERIES, WHICH GIVES NUMERICAL SOLUTION, WHICH USUALLY IS NOT EXACT. THUS TO GET THE EXACT SOLUTION, WE ARE USING COMBINATION OF ADOMAIN DECOMPOSITION METHOD AND MAHGOU'B TRANSFORM, WITH PADÉ'S APPROXIMATION.

2. Some preliminaries:

2.1 Mahgoub Transform[1,3].

we can take set A the function is defined in the form

$$A = \left\{ f: \left| f(t) \right| < p e^{\frac{|t|}{\varepsilon_i}} \text{ if } t \in (-1)^i \times [0, \infty), i = 1, 2; \varepsilon_i > 0 \right\}, \quad \dots (2.1)$$

The constant P must be finite number ε_1 & ε_2 may be finite or infinite.

Then the Mahgoub Transform define as,

$$M(f(t)) = H(u) = u \int_0^\infty f(t) e^{-ut} dt, \quad t \geq 0, \quad \varepsilon_1 \leq u \leq \varepsilon_2; \quad \dots (2.2)$$

2.2 Mahgoub Transform Of Derivative:

Let $f(t)$ be a function, and then the Mahgoub Transform of n^{th} order derivative of $f(t)$ with respect to t is given by

$$M[f^n(t)] = u^n H(u) - \sum_{k=0}^{n-1} u^{n-k} f^k(0) \quad \dots (2.3)$$

For $n = 1, 2, 3, \dots$ in equation (2.3) give Mahgoub Transform of first and second derivative of $f(t)$ w.r.t. " t ".

$$M[f'(t)] = uH(u) - uf(0)$$

$$M[f''(t)] = u^2H(u) - uf'(0) - u^2f(0)$$

2.2 Adomian Decomposition Method:

Adomian decomposition method is a semi analytical method for solving varied types of differential and integral equation, both linear and non-linear, and including partial differential equations [8]. This method was introduced by George Adomian in 1980s. The main advantage of this method is that it reduces the size of computation work and maintains the high accuracy of the numerical solution[9]. In ADM, a solution can be decomposed into an infinite series that converges rapidly into the exact solution. The linear and non-linear portion of the equation can be separated by ADM. The inversion of linear operator can be represented by the linear operator any given

condition is taken into consideration. The decomposition of a series is obtained by non linear portion which is called Adomian polynomials. By the using Adomian polynomials we can find a solution in the form of a series which can be determined by the recursive relationship.

Let the equation is $H = h$... (2.4)

where H is non-linear operator and h can be any function and value. Now H can be represented by N operator as follow, which is invertible

$$Ny + Ry + N^*y = h \quad \dots (2.5)$$

Here the liner operator 'R' represents the reminder of the linear portion and N^* is a non-linear operator representing the non-linear terms in H . Now, apply inverse operator N^{-1} on both side of the equation of (2.5) we have following equation

$$N^{-1}Ny = N^{-1}h - N^{-1} - RyN^{-1}N^*y \quad \dots (2.6)$$

Now equation (2.6) becomes

$$y(x) = g(x) - N^{-1}Ry - N^{-1}N^*y \quad \dots (2.7)$$

Where $g(x)$ represent the function obtained by integrating h and putting within the given boundary condition.

Let the unknown function $y(x)$ be an infinite series which is given by

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \quad 0 < n < \infty ; \quad \dots (2.8)$$

let take $y_0 = g(x)$... (2.9)

We can get other terms by using recursive relationship. The non -linear terms can be decomposed into a series that is called Adomian polynomial, A_n .

Now, the non-linear term can be written as

$$N^*y(x) = \sum_{n=0}^{\infty} A_n \quad \dots (2.10)$$

To find the value of A_n , we introduce a grouping parameter. Hence, A_n is given by

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} N^* y(\lambda) |_{\lambda=0} \quad \dots (2.11)$$

Thus the recursive relation obtained

$$y_0 = g(x) \\ y_{n+1} = N^{-1}Ry_n + N^{-1}A_n \quad \dots (2.12)$$

Thus the ADM gives a convergent series solution which is absolute and uniformly convergent.

3. Padés approximation:

To get a good approximation of a function like power series, we can use Padé approximation. Main advantage of Padé's approximation is we can approximate any continuous function on a closed interval to within arbitrary tolerance [10]. We have rational function (of a specified order) whose power series expansion agrees with a given power series to the highest possible order, or a function except the radius of convergence is sufficiently large to contain the domain $[a, b]$, over which the function is approximated, Now we are using rational function $f(x)$ of the form

$$\left[\frac{n}{m} \right] = \frac{p(x)}{q(x)} = \frac{\sum_{i=0}^n p_i x^i}{\sum_{i=0}^m 1 + q_i x^i} \quad a \leq x \leq b ;$$

Where $p(x)$ and $q(x)$ are two polynomials, with degree of m and n . Also their derivatives up to $m+n$ agree at $x=0$.

Assume that $f(x)$ is analytic and has Maclaurin series expansion given by

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots a_k x^k + \dots;$$

From the difference

$$f(x)q(x) - p(x) = y(x); \\ \left(\sum_{i=0}^{\infty} a_i x^i \right) \left(1 + \sum_{i=1}^m q_i x^i \right) - \left(\sum_{i=0}^n p_i x^i \right) = \left(\sum_{i=n+m+1}^{\infty} c_i x^i \right) \quad \dots (A)$$

Expanding (A) and equating the powers of x^i to zero for $i = 0, 1, 2, \dots, n+m$.

After that get $n+m+1$ linear equation, find the values of

$$p_1, p_2, \dots, p_n, q_1, q_2, \dots, q_m ;$$

3.1 Analysis of Admain Mahgoub Transform method (AMTM) :

Consider the Blasius equation

$$f'''(\phi) + \frac{1}{2} f(\phi) f''(\phi) = 0 \quad 0 \leq \phi \leq \infty ; \quad \dots (3.1)$$

Subject to the boundary conditions

$$f(0) = 0, f'(0) = 1, f'(\infty) = 0 \quad \dots (3.2)$$

Taking the M. T. of both sides of (3.1) we get

$$M\{f'''(\phi)\} = -\frac{1}{2}M\{f(\phi)f''(\phi)\} \quad \dots (3.3)$$

Applying the derivative property of Mahgoub Transform.

$$u^3M\{f(\phi)\} - u^2f(0) - uf'(0) - f''(0) = -\frac{1}{2}M\{f(\phi)f''(\phi)\}$$

$$u^3M\{f(\phi)\} = u^2f(0) + uf'(0) + f''(0) - \frac{1}{2}M\{f(\phi)f''(\phi)\} \quad \dots (3.4)$$

Using boundary conditions and taking $f''(0) = c$, we get

$$u^3M\{f(\phi)\} = u^2 \cdot 0 + u \cdot 1 + c - \frac{1}{2}M\{f(\phi)f''(\phi)\}$$

$$M\{f(\phi)\} = \frac{1}{u^2} + \frac{c}{u^3} - \frac{1}{2} \cdot \frac{1}{u^3}M\{f(\phi)f''(\phi)\} \quad \dots (3.5)$$

Taking inverse Mahgoub Transform both sides, we get

$$f(\phi) = t + \frac{ct^2}{2!} - \frac{1}{2}M^{-1}\left[\frac{1}{u^3}M\{f(\phi)f''(\phi)\}\right] \quad \dots (3.6)$$

Let

$$f(\phi) = \sum_{n=0}^{\infty} f_n(\phi) \quad \dots (3.7)$$

$$\sum_{n=0}^{\infty} f_n(\phi) = t + \frac{ct^2}{2!} - \frac{1}{2}M^{-1}\left[\frac{1}{u^3}M\left(\sum_{n=0}^{\infty} A_n\right)\right] \quad \dots (3.8)$$

Where

$$\sum_{n=0}^{\infty} A_n = f(\phi)f''(\phi) \quad \dots (3.9)$$

And the Adomain Polynomials A_n are calculated by using the formula

$$A_n = \frac{1}{n!} \frac{d^n}{d\theta^n} \left[\left(\sum_{i=0}^{\infty} \theta^i f_i \right) \left(\sum_{j=0}^{\infty} \theta^j f_j'' \right) \right]_{\theta=0} ; n = 0, 1, \dots \quad \dots (3.10)$$

The Adomain Polynomials are given by

$$A_0 = f_0(\phi)f_0''(\phi) \quad \dots (3.11)$$

$$A_1 = f_0(\phi)f_1''(\phi) + f_1(\phi)f_0''(\phi) \quad \dots (3.12)$$

$$A_2 = f_0(\phi)f_2''(\phi) + f_1(\phi)f_1''(\phi) + f_2(\phi)f_0''(\phi) \quad \dots (3.13)$$

$$A_3 = f_0(\phi)f_3''(\phi) + f_1(\phi)f_2''(\phi) + f_2(\phi)f_1''(\phi) + f_3(\phi)f_0''(\phi) \quad \dots (3.14)$$

$$A_4 = f_0(\phi)f_4''(\phi) + f_1(\phi)f_3''(\phi) + f_2(\phi)f_2''(\phi) + f_3(\phi)f_1''(\phi) + f_4(\phi)f_0''(\phi) \quad \dots (3.15)$$

We now take

$$f_0(\phi) = t + \frac{ct^2}{2!} ; f_0''(\phi) = 0 + c = c \quad \dots (3.16)$$

And the higher iterations of $f(\phi)$ are obtained from the recurrence relation,

$$f_{n+1}(\phi) = -\frac{1}{2}M^{-1}\left[\frac{1}{u^3}M\left(\sum_{n=0}^{\infty} A_n\right)\right] \quad n \geq 0 ; \quad \dots (3.17)$$

Thus we have

$$f_1(\phi) = -\frac{1}{48}c\phi^4 - \frac{1}{240}c^2\phi^5, \quad \dots (3.18)$$

$$f_2(\phi) = \frac{1}{960}c\phi^6 + \frac{11}{20160}c^2\phi^7 + \frac{11}{161280}c^3\phi^8, \quad \dots (3.19)$$

$$f_3(\phi) = -\frac{1}{21504}c\phi^8 - \frac{43}{967680}c^2\phi^9 - \frac{5}{387072}c^3\phi^{10} - \frac{1}{4257792}c^4\phi^{11}, \quad \dots (3.20)$$

Therefore the solution of Eq.(3.1) is in a series form given by

$$f(\phi) = \phi + \frac{1}{2}c\phi^2 - \frac{1}{48}c\phi^4 - \frac{1}{240}c^2\phi^5 + \frac{1}{960}c\phi^6 + \frac{11}{20160}c^2\phi^7 + \left(\frac{11c^3}{161280} - \frac{c}{21504}\right)\phi^8$$

$$- \frac{43}{967680}c^2\phi^9 - \frac{5}{387072}c^3\phi^{10} - \frac{11}{4257792}c^4\phi^{11} + \dots \quad \dots (3.21)$$

And so

$$f'(\phi) = 1 + c\phi - \frac{1}{12}c\phi^3 - \frac{1}{48}c^2\phi^4 + \frac{1}{160}c\phi^5 + \frac{11}{2880}c^2\phi^6 + \left(\frac{11}{20160}c^3 - \frac{1}{2688}\right)\phi^7 - \frac{43}{107520}c^2\phi^8$$

$$- \frac{5}{193536}c^3\phi^9 - \frac{11}{387072}c^4\phi^{10} + \dots \quad \dots (3.22)$$

Now using the condition at ∞ to find the value of the constant c that is $f(\infty) = 0$.

We cannot directly apply this condition on Eq. (3.22). The Pade's approximation is used to find rational function of Eq. (3.22), we have $f'(\phi)$ of degree 4.

Now we find when $n = m = 2$

$$\left[\frac{2}{2}\right] = \frac{1 + \frac{3}{4}c\phi + \left(\frac{1}{12} - \frac{1}{4}c^2\right)\phi^2}{1 - \frac{1}{4}c\phi + \frac{1}{12}\phi^2} \quad \dots (3.23)$$

Using the condition $f(\infty) = 0$ yields;

$$c = 0.5773502693 \quad \dots (3.24)$$

Now again by Padé's approximation when $n = m = 3$ to $f'(\phi)$ of degree 6 we obtain the value;

$$c = 0.5163977795 \quad \dots (3.25)$$

For Padé's approximation when $n = m = 4$ gives;

$$c = 0.5227030798 \quad \dots (3.26)$$

Mahgoub Transform comparison of numerical values of $c = f''(\phi)$

Padé Approximation	Adomain Mahgoub Transform Method (AMTM)	Adomain Sumudu Transform Method (ASTM)
$\left[\frac{2}{2}\right]$	0.5773502693	0.5773502693
$\left[\frac{3}{3}\right]$	0.5163977795	0.5163977795
$\left[\frac{4}{4}\right]$	0.5227030798	0.5227030798

Table -1

(comparison between Adomain Mahgoub Transform Method and Adomain Sumudu Transform Method)

4. Conclusion and Discussion: Padé's approximation has been demonstrated. The solution of Blasius equation by combination of the Mahgoub Transform and Adomain Decomposition Method are introduced. It is nearest from the solution of the Blasius equation. The result confirms that the Mahgoub Transform technique is a simple and powerful tool. We compare the result obtained with Adomain Mahgoub Transform and Adomain Sumudu Transform and see that they are the same. It is thus proved that the authenticity of Adomain Mahgoub Transform is reliable.

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