

Exact Solutions for a Compound KdV-Burgers Equation with Forcing Term

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Abstract

The exact solutions of a compound KdV-Burgers equation with forcing term are obtained by using a homogeneous balance method. Moreover finding the solutions for compound KdV, mKdV-Burgers, KdV-Burgers, mKdV and KdV equations with forcing term. These are the particular important cases of that equation.

Keywords: Compound KdV-Burgers equation with forcing term; Homogeneous balance method; Exact solutions.

1. INTRODUCTION

In the present paper we consider the compound KdV-Burgers equation with forcing term,

$$u_t + puu_x + qu^2u_x + ru_{xx} - su_{xxx} = f(t), \quad (1.1)$$

where $f(t)$ is an external forcing function varying with time variable t and p, q, r and s are constants, which can be involving to the construction of the KdV, mKdV and Burgers equations with forcing term, such as the nonlinear, dispersion and dissipation effects. Eq.(1.1) contains the following particular important cases.

(1) $p \neq 0, q \neq 0, r = 0, s \neq 0$: (1.1) becomes compound KdV equation with forcing term

$$u_t + puu_x + qu^2u_x - su_{xxx} = f(t), \quad (1.2)$$

(2) $p = 0, q \neq 0, r \neq 0, s \neq 0$: (1.1) becomes mKdV-Burgers equation with forcing term

$$u_t + qu^2u_x + ru_{xx} - su_{xxx} = f(t), \quad (1.3)$$

(3) $p \neq 0, q = 0, r \neq 0, s \neq 0$: (1.1) becomes KdV-Burgers equation with forcing term

$$u_t + puu_x + ru_{xx} - su_{xxx} = f(t), \quad (1.4)$$

(4) if $r = 0$ in (1.3) and (1.4), then we obtain the mKdV equation with forcing term

$$u_t + qu^2u_x - su_{xxx} = f(t), \quad (1.5)$$

and the KdV equation with forcing term

$$u_t + puu_x - su_{xxx} = f(t), \quad (1.6)$$

respectively.

It is well known that each of the KdV, mKdV and Burgers equation with forcing term is exactly solvable and many studies of these equations have already been undertaken. We are the first report studies of the compound KdV-Burgers equation with forcing term which includes the KdV-Burgers and mKdV-Burgers equation with forcing term by using homogeneous balance method [1] , To see there are some studies for the KdV-Burgers equation with forcing term [2]-[6] .

Outline of this paper - in section 2 , Analysis of the homogeneous balance method, in section 3, the exact solutions of Eq.(1.1) will be found by use of this method [1], and exact solutions of Eqs.(1.2) ,(1.3) and (1.5) as the particular cases of the solutions for Eq.(1.1), are obtained. But the solutions for Eq.(1.4) and (1.6) are cannot be obtained from the solution of Eq. (1.1) because solution forms are different to each other, therefore in section 4, we start from Eq. (1.4) to use of this method [1], to get its exact solutions, in section 5, conclusion our results.

2. ANALYSIS OF THE METHOD

Now we describe that

1. What is the homogeneous balance method and
2. How to use it to look for special exact solutions of some nonlinear equations in mathematical physics.

Suppose Given a partial differential equation, say in two variables,

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots \dots \dots) = 0. \quad (2.1)$$

Where P is an general a polynomial function of its arguments.

Eqs.(1.1)-(1.6) are all different to the type of Eq.(2.1). So we use the transformation

$$u(x, t) = v(x, t) + \int f(t) dt \quad (2.2)$$

To Eqs.(1.1)-(1.6), Then Eqns.(1.1)-(1.6) becomes,

$$v_t + p(v + \int f(t) dt)v_x + q(v + \int f(t) dt)^2 v_x + rv_{xx} - sv_{xxx} = 0, \quad (2.3)$$

$$v_t + p(v + \int f(t) dt)v_x + q(v + \int f(t) dt)^2 v_x - sv_{xxx} = 0, \quad (2.4)$$

$$v_t + q(v + \int f(t) dt)^2 v_x + rv_{xx} - sv_{xxx} = 0, \quad (2.5)$$

$$v_t + p(v + \int f(t) dt)v_x + rv_{xx} - sv_{xxx} = 0, \quad (2.6)$$

$$v_t + q(v + \int f(t) dt)^2 v_x - sv_{xxx} = 0, \quad (2.7)$$

$$v_t + p(v + \int f(t) dt)v_x - sv_{xxx} = 0, \quad (2.8)$$

Therefore, Eqns.(2.3)-(2.8) are all same type of Eqn.(2.1). A function $w = w(x, t)$ is called a quasi solution of Eq.(2.1), if there exists a function $f = f(w)$ of one variable only so that a suitable linear combination of following functions,

$$1, f(w), f(w)_x, f(w)_t, f(w)_{xx}, f(w)_{xt}, f(w)_{tt}, \dots \dots \quad (2.9)$$

is actually a solution of Eq.(2.1). Here we will show how to find $f(w)$, the quasi solution $w = w(x, t)$ and a suitable linear combination of the functions in (2.9), and then to obtain special exact solutions of each equation in (2.3)-(2.8). This method of looking for special solutions of a nonlinear equation in mathematical physics is called the homogenous balance method which consists of four steps:

First step: choose a suitable linear combination of the functions in (2.9), maybe its coefficients to be determined, so that the highest nonlinear terms and the highest order partial derivative terms in the given equation are both transformed into the polynomials with a highest equality degree in partial derivatives of $w(x, t)$ in spite of $f(w)$ and its various derivatives. "The highest equality degree" here is quite essential.

Taking the KdV-Burgers equation with forcing term (2.6) as an example to explain explicitly the first step of the method assume that a linear combination of the functions in (2.9) is of the form

$$\begin{aligned} v &= \frac{\partial^{m+n}}{\partial x^m \partial t^n} + \text{all partial derivative terms with lower than } (m+n) \text{ order of } f(w) \\ &= f^{(m+n)} w_x^m w_t^n \\ &+ \text{all terms with lower than } (m+n) \text{ degree in various partial derivatives of } w(x, t), \end{aligned} \quad (2.10)$$

Where $m \geq 0, n \geq 0$ are integers to be determined.

The nonlinear term in Eq.(2.6) is transformed into

$$\begin{aligned} vv_x &= f^{(m+n)} f^{(m+n+1)} w_x^{(2m+1)} w_t^{2n} \\ &+ \text{all terms with lower than } 2(m+n)+1 \text{ degree in various partial derivatives of } w(x, t). \end{aligned} \quad (2.11)$$

The highest order partial derivative term in Eq.(2.6) is transformed into

$$\begin{aligned} v_{xxx} &= f^{(m+n+3)} w_x^{(m+3)} w_t^n \\ &+ \text{all terms with lower than } (m+n+3) \text{ degree in various partial derivatives of } w(x, t). \end{aligned} \quad (2.12)$$

Requiring the highest degrees in partial derivatives of $w(x, t)$ in (2.11) and (2.12) are equal (i.e. nonlinear and dispersive effects are partially balanced) yields

$$2m + 1 = m + 3, \quad 2n = n, \quad (2.13)$$

Which has a non-negative integer solution: $m = 2, n = 0$, therefore we can choose the linear combination as follows,

$$v = f_{xx} + af_x + b = f''w_x^2 + f'w_x + b, \quad (2.14)$$

Which is the origin of (4.1) in section 4 of the present paper.

Second step: Substituting the linear combination chosen in the first step into Eq.(2.1), collecting all terms with the highest degree of derivatives of $w(x, t)$ and setting its coefficient to zero (we call that making a partial balance between the highest nonlinear terms and highest order partial derivative terms in Eq.(2.1)), we obtain an ordinary differential equation for $f(w)$ and then solve it, in most cases $f(w)$ is a logarithm function (in the KdV-Burgers case, w_x^5 is the highest degree in partial derivatives of $w(x, t)$ in the nonlinear and highest order partial derivatives terms, the ODE and its solution are in (4.7) and (4.8) respectively, in section 4 of the present paper).

Third step: Starting from the ODE and its solution obtained above, the nonlinear terms of various derivatives of $f(w)$ in the expression obtained in the second step can be replaced by the corresponding higher order derivatives of $f(w)$ (in the Kdv-Burgers case, the results are in (4.9)). After doing this, collecting all terms with the same order derivatives of $f(w)$ and setting the coefficient of each order derivative of $f(w)$ to zero respectively, we obtain a set of equations for $w(x, t)$, the left hand sides of these equations are the k degree homogeneous functions in various derivatives of $w(x, t)$, where k is the order of f^k (in the Kdv-Burgers case see (4.10)). In view of the homogeneous property of these equations we can expect that $w(x, t)$ is an exponential function with some variable functions to be determined. If there exists a solution for these nonlinear algebraic equations, then $w(x, t)$ and the coefficients of the linear combination chosen in the first step can be determined (in the Kdv-Burgers case, see (4.11)-(4.13)).

Fourth step: Substituting $f(w)$ and $w(x, t)$ as well as some constants obtained in the second and third steps into the combination chosen in the first step, after doing some calculations, we then obtain an exact solution of Eq.(2.1) (in the Kdv-Burgers case, see the result in (4.17)).

3. COMPOUND KDV-BURGERS EQUATION WITH FORCING TERM

In this section the exact solution of the compound KdV-Burgers equation with forcing term (1.1) and also some particular important cases (1.2),(1.3) and (1.5) will be found by homogeneous balance method.

In order that the nonlinear term qv^2v_x and the third order derivative term $-sv_{xxx}$ in Eq.(2.3) can be partially balanced, we suppose that the solution of (2.3) is of the form

$$v(x, t) = af_x(w) + b = af'_x w_x + d, \quad w = w(x, t) \quad (3.1)$$

Where the function f and the function w as well as the constants a and $d = ab$ are to be determined. From (2.1) one obtains

$$v_t = af''w_x w_t + af'_x w_{xt}, \quad (3.2)$$

$$p v v_x = pa^2 (f' f'' w_x^3 + f'^2 w_x w_{xx} + bf'' w_x^2 + bf' w_{xx}), \quad (3.3)$$

$$p \left\{ \int f(t) dt \right\} v_x = pa \left\{ \int f(t) dt \right\} (f'' w_x^2 + f' w_{xx}), \quad (3.4)$$

$$q v^2 v_x = qa^3 (f'^2 f'' w_x^4 + f'^3 w_x^2 w_{xx} + 2bf' f'' w_x^3 + 2bf'^2 w_x w_{xx} + b^2 f'' w_x^2 + b^2 f' w_{xx}) \quad (3.5)$$

$$q \left(\int f(t) dt \right)^2 v_x = qa \left(\int f(t) dt \right)^2 (f'' w_x^2 + f' w_{xx}) \quad (3.6)$$

$$2q \left(\int f(t) dt \right) v v_x = 2qa^2 \left(\int f(t) dt \right) (f' f'' w_x^3 + bf'' w_x^2 + f'^2 w_x w_{xx} + bf' w_{xx}) \quad (3.7)$$

$$r v_{xx} = ra(f^{(3)} w_x^3 + 3f'' w_x w_{xx} + f' w_{xxx}) \quad (3.8)$$

$$-s v_{xxx} = -sa(f^{(4)} w_x^4 + 6f^{(3)} w_x^2 w_{xx} + 4f'' w_x w_{xxx} + 3f'' w_{xx}^2 + f' w_{xxxx}). \quad (3.9)$$

First collecting the terms with w_x^4 in (3.5) and (3.9), and setting its coefficient to zero, we obtain an ordinary differential equation for $f(w)$

$$sf^{(4)} - qa^2 f'^2 f'' = 0 \quad (3.10)$$

Specifically, we assume that

$$qs > 0, \quad a > 0, \quad (3.11)$$

Then the ODE (3.10) has two solutions

$$f = \pm \frac{1}{a} \sqrt{\frac{6s}{q}} \ln(w), \quad (3.12)$$

Thereby

$$f'f'' = \mp \frac{1}{2a} \sqrt{\frac{6s}{q}} f^{(3)}, \quad f'^3 = \frac{3s}{qa^2} f^{(3)}, \quad f'^2 = \mp \frac{1}{a} \sqrt{\frac{6s}{q}} f''. \quad (3.13)$$

Substituting (3.2)-(3.9) into the left hand side of Eq. (2.3) and using (3.10)-(3.13), we obtain

$$\begin{aligned} & v_t + p(v + \int f(t) dt)v_x + q(v + \int f(t) dt)^2 v_x + rv_{xx} - sv_{xxx} \\ &= [\{ra \mp \frac{pa}{2} \sqrt{\frac{6s}{q}} \mp qa \sqrt{\frac{6s}{q}} (ab + \int f(t) dt)\} w_x^3 - 3sa w_x^2 w_{xx}] f^{(3)} \\ &+ [aw_x w_t + (3ra \mp \{pa + 2qa^2 b + 2qa \int f(t) dt\} \sqrt{\frac{6s}{q}}) w_x w_{xx} \\ &+ (pa^2 b + qa^3 b^2 + a \int f(t) dt \{p + q(\int f(t) dt) + 2qab\}) w_x^2 - 3sa w_{xx}^2 - 4sa w_x w_{xxx}] f^{(2)} + \\ &[aw_{xt} + (pa^2 b + qa^3 b^2 + a \int f(t) dt \{p + q(\int f(t) dt) + 2qab\} + ra w_{xxx} - sa w_{xxxx})] f'. \end{aligned} \quad (3.14)$$

Setting the coefficients of $f^{(3)}$, f'' and f' in (3.14) to zero respectively, we obtain a set of equations for $w(x, t)$

$$\begin{aligned} & \{r \mp \frac{p}{2} \sqrt{\frac{6s}{q}} \mp q \sqrt{\frac{6s}{q}} (ab + \int f(t) dt)\} w_x^3 - 3s w_x^2 w_{xx} = 0, \\ & w_x w_t + \{3r \mp (p + 2qab + 2q \int f(t) dt) \sqrt{\frac{6s}{q}}\} w_x w_{xx} + (pab + qa^2 b^2 + \int f(t) dt \{p + \\ & \quad q(\int f(t) dt) + 2qab\}) w_x^2 - 3s w_{xx}^2 - 4s w_x w_{xxx} = 0, \\ & w_{xt} + (pab + qa^2 b^2 + \int f(t) dt \{p + q(\int f(t) dt) + 2qab\}) w_{xx} + r w_{xxx} - s w_{xxxx} = 0. \end{aligned} \quad (3.15)$$

From the discussion above we come to the conclusion that if we take f as in (3.12) and $w(x, t)$ satisfies Eq.(3.15), then (3.1) is actually a solution of Eq.(2.3).

To solve Eq.(3.15), we can suppose that the solution of Eqs.(3.15) is of the form

$$w(x, t) = 1 + \exp(\alpha(t)x + \beta(t)t + \theta_0), \quad (3.16)$$

Where $\alpha(t)$ and $\beta(t)$ are variable functions to be determined, and θ_0 is an arbitrary constant.

Substituting (3.16) into (3.15), we find that (3.16) satisfies (3.15) provided that a , b , $\alpha(t)$ and $\beta(t)$ satisfy the following conditions

$$\begin{aligned} & \left(r \mp \frac{p}{2} \sqrt{\frac{6s}{q}} \mp q \sqrt{\frac{6s}{q}} (ab + \int f(t) dt) \right) \alpha^3 - 3s \alpha^4 = 0, \\ & \alpha(\alpha'x + \beta't + \beta) + \left(3r \mp (p + 2qab + 2q \int f(t) dt) \sqrt{\frac{6s}{q}} \right) \alpha^3 + (pab + qa^2 b^2 + \int f(t) dt (p + \\ & \quad q(\int f(t) dt) + 2qab) \alpha^2 - 7s \alpha^4 = 0, \end{aligned}$$

$$\alpha(\alpha'x + \beta't + \beta) + \alpha' + \left(pab + qa^2b^2 + \int f(t) dt (p + q \int f(t) dt) + 2qab \right) \alpha^2 + r\alpha^3 - s\alpha^4 = 0. \quad (3.17)$$

Solving the algebraic equations (3.17) we obtain

$$a > 0, \quad b \text{ are arbitrary constants}, \quad \alpha(t) = \alpha_{\mp}(t), \quad \beta = \beta_{\mp}(t), \quad (3.18a)$$

Where

$$\alpha_{\mp}(t) = c, \quad \beta_{\mp}(t) = \frac{-c^2}{t} \int \left[\frac{\int f(t) dt [p+q \int f(t) dt + 2qab]}{t} \right] dt - \frac{\log t \quad c^2}{t} [pab + qa^2b^2 + rc - sc^2] \quad (3.19)$$

Since we have found $f(w)$ and $w(x, t)$, by substituting (3.12) and (3.16) with (3.18a) and (3.19) into (3.1) and using the equality

$$\frac{\exp \theta}{1 + \exp \theta} = \frac{1}{2} \tanh \frac{1}{2} \theta + \frac{1}{2},$$

We obtain a pair of exact solutions of Eq. (2.3)

$$v(x, t) = \pm \sqrt{\frac{6s}{q}} \frac{c \exp \theta_{\mp}}{1 + \exp \theta_{\mp}} + d = \pm \frac{c}{2} \sqrt{\frac{6s}{q}} \left(\tanh \frac{1}{2} \theta_{\pm} + 1 \right) + d, \quad (3.20)$$

Where

$$\theta_{\mp} = \alpha_{\mp}(t)x + \beta_{\mp}(t)t + \theta_0, \quad \theta_0 = \text{an arbitrary constant} \quad (3.21)$$

Now, a pair of exact solutions of Eq.(1.1) by using (2.2), (3.20) and (3.21) we get

$$u(x, t) = \pm \frac{c}{2} \sqrt{\frac{6s}{q}} \left(\tanh \frac{1}{2} \left(cx + \left[\frac{-c^2}{t} \int \left[\frac{\int f(t) dt [p+q \int f(t) dt + 2qab]}{t} \right] dt - \frac{\log t \quad c^2}{t} [pab + qa^2b^2 + rc - sc^2] \right] t + \theta_0 \right) + 1 \right) + d + \int f(t) dt. \quad (3.22)$$

Where c,b,d and a are arbitrary constants.

Now consider two special cases of the compound KdV-Burgers equation with forcing term:

(1). If $r = 0$ in (2.3), then the algebraic equations (3.17) have solutions

$$a > 0 \text{ and } b \text{ are arbitrary constants}, \quad \alpha(t) = \alpha_{1\mp}(t), \quad \beta = \beta_{1\mp}(t), \quad (3.18b)$$

Where

$$\alpha_{1\mp}(t) = c, \quad \beta_{1\mp}(t) = \frac{-c^2}{t} \int \left[\frac{\int f(t) dt [p+q \int f(t) dt + 2qab]}{t} \right] dt - \frac{\log t \quad c^2}{t} [pab + qa^2b^2 - sc^2] \quad (3.23)$$

And a pair of exact solutions of the compound KdV equation with forcing term (2.4) are obtained by substituting (3.12) and (3.16) with (3.18b) and (3.23) into (3.1) namely

$$v(x, t) = \pm \sqrt{\frac{6s}{q}} \frac{c \exp \theta_{1\mp}}{1 + \exp \theta_{1\mp}} + d = \pm \frac{c}{2} \sqrt{\frac{6s}{q}} \left(\tanh \frac{1}{2} \theta_{1\mp} + 1 \right) + d, \quad (3.24)$$

Where

$$\theta_{1\mp} = \alpha_{1\mp}(t)x + \beta_{1\mp}(t)t + \theta_0, \quad \theta_0 = \text{an arbitrary constant} \quad (3.25)$$

Now , a pair of exact solutions of Eq.(1.2) by using (2.2) , (3.23), (3.24) and (3.25) we get

$$u(x, t) = \pm \frac{c}{2} \sqrt{\frac{6s}{q}} \left(\tanh \frac{1}{2} \left(cx + \left[\frac{-c^2}{t} \int \left[\frac{\int f(t) dt [p+q \int f(t) dt + 2qab]}{t} \right] dt - \frac{\log t}{t} [pab + qa^2b^2 - sc^2] \right) t + \theta_0 \right) + 1 \right) + d + \int f(t) dt. \quad (3.26)$$

Where c, b, d and a are arbitrary constants.

If $p = 0$ and in (3.24), then we obtain a pair of exact solutions of (2.7) is,

$$v(x, t) = \pm \sqrt{\frac{6s}{q}} \frac{c \exp \theta_{2\mp}}{1 + \exp \theta_{2\mp}} + d = \pm \frac{c}{2} \sqrt{\frac{6s}{q}} \left(\tanh \frac{1}{2} \theta_{2\mp} + 1 \right) + d, \quad (3.27)$$

Where

$$\theta_{2\mp} = \alpha_{2\mp}(t)x + \beta_{2\mp}(t)t + \theta_0, \quad \theta_0 = \text{an arbitrary constant}$$

$$\alpha_{2\mp}(t) = c, \quad \beta_{2\mp}(t) = \frac{-c^2}{t} \int \left[\frac{\int f(t) dt [q \int f(t) dt + 2qab]}{t} \right] dt - \frac{\log t}{t} [qa^2b^2 - sc^2] \quad (3.28)$$

Now , a pair of exact solutions of Eq.(1.5) by using (2.2) , (3.27) and (3.28) , we get

$$u(x, t) = \pm \frac{c}{2} \sqrt{\frac{6s}{q}} \left(\tanh \frac{1}{2} \left(cx + \left[\frac{-c^2}{t} \int \left[\frac{\int f(t) dt [q \int f(t) dt + 2qab]}{t} \right] dt - \frac{\log t}{t} [qa^2b^2 - sc^2] \right) t + \theta_0 \right) + 1 \right) + d + \int f(t) dt. \quad (3.29)$$

Where c, b, d and a are arbitrary constants..

(2) . If $p = 0$ in (2.3), then algebraic equations (3.17) have solutions

$$a > 0, \quad b \text{ are arbitrary constants}, \quad \alpha(t) = \alpha_{3\mp}(t), \quad \beta = \beta_{3\mp}(t), \quad (3.18c)$$

Where

$$\alpha_{3\mp}(t) = c, \quad \beta_{3\mp}(t) = \frac{-c^2}{t} \int \left[\frac{\int f(t) dt [q \int f(t) dt + 2qab]}{t} \right] dt - \frac{\log t}{t} [qa^2b^2 + rc - sc^2] \quad (3.30)$$

And a pair of exact solutions for the mKdv-Burgers equation with forcing term (2.5) are obtained by substituting (3.12) and (3.16) with (3.18c) and (3.30) into (3.1), namely

$$v(x, t) = \pm \frac{c}{2} \sqrt{\frac{6s}{q}} \left(\tanh \frac{1}{2} \theta_{3\mp} + 1 \right) + d, \quad (3.31).$$

Where

$$\theta_{3\mp} = \alpha_{3\mp}(t)x + \beta_{3\mp}(t)t + \theta_0 \quad (3.32)$$

Now , a pair of exact solutions of Eq.(1.3) by using (2.2) ,(3.30), (3.31) and (3.32), we get

$$u(x,t) = \pm \frac{c}{2} \sqrt{\frac{6s}{q}} \left(\tanh \frac{1}{2} \left(cx + \left[\frac{-c^2}{t} \int \left[\frac{\int f(t) dt [q \int f(t) dt + 2qab]}{t} \right] dt - \frac{\log t}{t} [qa^2b^2 + rc - sc^2] \right) t + \theta_0 \right) + 1 \right) + d + \int f(t) dt. \quad (3.33)$$

Where c, b, d and a are arbitrary constants..

We cannot obtain the solution of the KdV-Burgers equation with forcing term (1.4) and KdV equation with forcing term (1.6) from (3.22) as $q = 0$ and $q = r = 0$ because solution form is different to the above type of equations solution form.

In the next section, we will discuss the KdV-Burgers equation with forcing term and KdV equation with forcing term.

4. KDV-BURGERS EQUATION WITH FORCING TERM

In this section the exact solution of the KdV-Burgers equation with forcing term (1.4) and thus the particular case - KdV equation with forcing term (1.6) also will be found by homogeneous balance method.

In order that the nonlinear term pvv_x and the third order derivative term $-sv_{xxx}$ in Eq.(2.6) can be partially balanced, we suppose that the solution of (2.6) is of the form

$$v(x,t) = f_{xx} + af_x(w) + b = f''w_x^2 + f'w_{xx} + af'w_x + b, \quad w = w(x,t) \quad (4.1)$$

Where the function f and the function w as well as the constants a and b are to be determined. From (4.1) one obtains

$$v_t = f^{(3)}w_x^2w_t + f''(aw_xw_t + 2w_xw_{xt} + w_{xx}w_t) + f'(aw_{xt} + w_{xxt}), \quad (4.2)$$

$$pvv_x = p \left[f''f^{(3)}w_x^5 + f''^2(3w_x^3w_{xx} + aw_x^4) + f'f^{(3)}(w_x^3w_{xx} + aw_x^4) + bf^{(3)}w_x^3 + f'f''(a^2w_x^3 + w_x^2w_{xxx} + 5aw_x^2w_{xx} + 3w_xw_{xx}^2) + f''(abw_x^2 + 3bw_xw_{xx}) + f'^2(a^2w_xw_{xx} + w_{xx}w_{xxx} + aw_{xx}^2 + aw_xw_{xxx}) + f'(bw_{xxx} + abw_{xx}) \right], \quad (4.3)$$

$$p \{ \int f(t) dt \} v_x = pa \{ \int f(t) dt \} (f'''w_x^3 + 3f''w_xw_{xx} + f'w_{xxx} + af''w_x^2 + af'w_{xx}), \quad (4.4)$$

$$r v_{xx} = r \left[f^{(4)}w_x^4 + f^{(3)}(6w_x^2w_{xx} + aw_x^3) + f''(3w_{xx}^2 + 3aw_xw_{xx} + 4w_xw_{xxx}) + f'(w_{xxxx} + aw_{xxx}) \right], \quad (4.5)$$

$$-s v_{xxx} = -s \left[f^{(5)}w_x^5 + f^{(4)}(10w_x^3w_{xx} + aw_x^4) + f^{(3)}(10w_x^2w_{xxx} + 6aw_x^2w_{xx} + 15w_xw_{xx}^2) + f''(3aw_{xx}^2 + 5w_xw_{xxxx} + 10w_{xx}w_{xxx} + 4aw_xw_{xxx}) + f'(w_{xxxxx} + aw_{xxxx}) \right] \quad (4.6)$$

Collecting all terms with w_x^5 in (4.3) and (4.6), and setting the coefficients of w_x^5 to zero, we obtain ordinary differential equation

$$sf^{(5)} - pf''f^{(3)} = 0, \quad (4.7)$$

Which has a solution

$$f = -\frac{12s}{p} \ln(w), \quad (4.8)$$

Thereby

$$f'f'' = \frac{6s}{p} f^{(3)}, \quad f'^2 = \frac{12s}{p} f^{(2)}, \quad f''^2 = \frac{2s}{p} f^{(4)}, \quad f'f''' = \frac{4s}{p} f^{(4)}. \quad (4.9)$$

Substituting (4.2)-(4.6) into the left hand side of Eq.(2.6), and using (4.9), we have

$$\begin{aligned} v_t + p \left(v + \int f(t) dt \right) v_x + r v_{xx} - s v_{xxx} \\ = f^{(4)}(r + 5sa)w_x^4 \\ + f^{(3)} \left[w_x^2 w_t + \left(pb + ra + 6sa^2 + p \int f(t) dt \right) w_x^3 - 4s w_x^2 w_{xxx} + (24as + 6r)w_x^2 w_{xx} \right. \\ \left. + 3s w_x w_{xx}^2 \right] \\ + f'' \left[2 w_x w_{xt} + w_{xx} w_t + a w_x w_t + (3pb + 3ra + 12sa^2 + 3p \left(\int f(t) dt \right) w_x w_{xx} \right. \\ \left. + pa \left(b + \int f(t) dt \right) w_x^2 + 2s w_{xx} w_{xxx} + (9sa + 3r)w_{xx}^2 + (8sa + 4r)w_x w_{xxx} \right. \\ \left. - 5s w_x w_{xxxx} \right] \\ + f' \left[-s w_{xxxx} + (r - sa) w_{xxxx} + (ra + pb + p \left(\int f(t) dt \right) w_{xxx} + w_{xxt} \right. \\ \left. + pa \left(b + \int f(t) dt \right) w_{xx} + a w_{xt} \right]. \end{aligned} \quad (4.10)$$

Now we take

$$a = \frac{-r}{5s} \quad (4.11)$$

And assume

$$w = 1 + \exp(\alpha(t)x + \beta(t)t), \quad (4.12)$$

With $\alpha(t)$ and $\beta(t)$ are variable functions to be determined. Substituting (4.11) and (4.12) into (4.10), we find that the right hand side of (4.10) vanishes provided that b , $\alpha(t)$ and $\beta(t)$ satisfy the following algebraic equations

$$\alpha^2(\alpha'x + \beta't + \beta) + \left(pb + \frac{r^2}{25s} + p \left(\int f(t) dt \right) \right) \alpha^3 + \frac{6r}{5} \alpha^4 - s \alpha^5 = 0,$$

$$\alpha^2 \left(3(\alpha'x + \beta't + \beta) - \frac{pr}{5s} \int f(t) dt - \frac{prb}{5s} \right) + \alpha \left(2\alpha' - \frac{r}{5s} (\alpha'x + \beta't + \beta) \right) + \frac{18r}{5} \alpha^4 +$$

$$\left(3p \left(\int f(t) dt + b \right) - \frac{3r^2}{25s} \right) \alpha^3 - 3s \alpha^5 = 0,$$

$$-\frac{\alpha' r}{5s} + \alpha \left(\frac{-r}{5s} (\alpha' x + \beta' t + \beta) + 2\alpha' \right) + \alpha^2 \left((\alpha' x + \beta' t + \beta) + \frac{-prb}{5s} + \left(\frac{-pr}{5s} \right) (\int f(t) dt) \right) + \alpha^3 \left(\frac{-r^2}{5s} + pb + p(\int f(t) dt) \right) + \alpha^4 \left(\frac{6r}{5} \right) - \alpha^5 s = 0 \quad (4.13)$$

Solving the algebraic equations (4.13) we obtain

$$\alpha(t) = c, \beta(t) = \frac{-pc}{t} \int \left[\frac{f(t) dt}{t} \right] dt - \frac{\log t}{5t} [5pcb + 6c^2 r] \quad (4.14)$$

Since we have found $f(w)$ and $w(x,t)$, by substituting (4.8), (4.12) and (4.14) into (4.1) and using the equality

$$\frac{\exp \theta}{1 + \exp \theta} = \frac{1}{2} \tanh \frac{1}{2} \theta + \frac{1}{2},$$

We obtain a exact solution of Eq. (2.6) is,

$$\begin{aligned} v(x,t) &= \frac{12s c^2}{p} \left[\left(\frac{\exp \theta}{1 + \exp \theta} \right)^2 - \frac{\exp \theta}{1 + \exp \theta} \right] + \frac{12rc}{5p} \left[\frac{\exp \theta}{1 + \exp \theta} \right] + b \\ &= -\frac{3s c^2}{p} \operatorname{sech}^2 \frac{1}{2} \theta + \frac{6rc}{5p} \tanh \frac{1}{2} \theta + \frac{6rc}{5p} + b, \end{aligned} \quad (4.15)$$

Where

$$\theta = \alpha(t)x + \beta(t)t + \theta_0, \quad \theta_0 = \text{an arbitrary constant} \quad (4.16)$$

Now, exact solutions of Eq.(1.4) by using (2.2), (4.14), (4.15) and (4.16), we get

$$\begin{aligned} u(x,t) &= -\frac{3s c^2}{p} \operatorname{sech}^2 \frac{1}{2} \left[cx + \left[\frac{-pc}{t} \int \left[\frac{f(t) dt}{t} \right] dt - \frac{\log t}{5t} [5pcb + 6c^2 r] \right] t + \theta_0 \right] + \\ &\quad \frac{6rc}{5p} \tanh \frac{1}{2} \left[cx + \left[\frac{-pc}{t} \int \left[\frac{f(t) dt}{t} \right] dt - \frac{\log t}{5t} [5pcb + 6c^2 r] \right] t + \theta_0 \right] + \frac{6rc}{5p} + b + \int f(t) dt. \end{aligned} \quad (4.17)$$

Where b and c are arbitrary constants.

It should be noted that we cannot obtain the solution of the KdV equation (1.6) from (4.17) as $r = 0$. however, we can still suppose that the solution of KdV equation (1.6) is of the form (4.1), then the results of (4.2)-(4.13) also hold as $r = 0$ and therefore $a = 0$.

From (4.17) as $r = 0$ we obtain the exact solution of (1.6) is

$$u(x,t) = -\frac{3s c^2}{p} \operatorname{sech}^2 \frac{1}{2} \left[cx + \left[\frac{-pc}{t} \int \left[\frac{f(t) dt}{t} \right] dt - \frac{\log t}{5t} [5pcb] \right] t + \theta_0 \right] + b + \int f(t) dt. \quad (4.18)$$

Where b and c are arbitrary constants.

5. CONCLUSION

In this paper we apply the homogeneous balance method to compound KdV-Burgers equation, successfully obtained special exact solutions to that equation. Also use of the homogeneous balance method, special exact solutions of many other typical nonlinear equations in mathematical physics, such as the Burgers equation,

Benjamin-Bona-Mathony equation, Boussinesq equation, Kuramoto – Sivashinsky equation with forcing term and so on, can be obtained.

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