Common Fixed Point on Cone Rectangular Metric Space

Jigmi Dorjee Bhutia, Kalishankar Tiwary
Dept of Mathematics,
Cooch Behar College, Cooch Behar, West Bengal, India.
Dept of Mathematics,
Raiganj University, Raiganj, West Bengal, India

Abstract:
In this paper we obtain some results on fixed point theorem in cone rectangular metric space which extends some known results that are already proved in [8,14].

Keywords: *Cone rectangular metric space, coincidence point, fixed point, Weakly Compatible mappings.*

Introduction:

Jungck [1] introduced the concept of commuting maps and proved the results that generalize the Banach contraction principle. Sessa [2] further gave the idea of weakly commuting maps thereby generalizing the commuting maps. Jungck [3] again extended this concept to compatible mappings. In 1998 [4] Jungck and Rhoades introduced the notion of weakly compatible maps. Compatible maps are weakly compatible but the converse is not true.

It is during the year 2007 when Huang and Zhang [5] introduced the concept of cone metric space by replacing the range set of non negative real numbers of the metric *d* by the ordered Banach space. The existence of a common fixed point in cone metric space has been considered recently in [6-22] and references therein.

In 2000 Branciari [6] introduced a class of generalized metric spaces by replacing triangular inequality by similar ones which involve four or more points instead of three and improved Banach contraction mapping principle.

Recently, Azam et.al [7] introduced the notion of cone rectangular metric space and proved Banach contraction mapping principle in a cone rectangular metric space setting.

Preliminaries

We recall some definitions and other results that will be needed in the sequel.

Definition 1.1 [5]: Let E be a real Banach space and $P \subset E$. Then P is said to be a cone if it satisfies the following condition:

i) P is a non-empty closed subset of E and $P \neq \{\theta\}$.

ii) If $x, y \in P$, *and* $a, b \in R$, $a \geq 0$, $b \geq 0$, then $ax + by \in P$.

iii) If $x \in P$ *and* $-x \in P$, then $x = \theta$.

Cone induces a Partial order relation

We can define a partial order relation on E with respect to the cone P in the following way $x \leq y$ if and only if $y - x \in P$. Also $x << y$ if and only if $y - x \in \operatorname{int} P$ and $x < y$ implies $x \leq y$ but $x \neq y$. If $\operatorname{int} P \neq \phi$ then the cone is a solid cone.

Definition 1.2[5]: Let X be a set and $d : X \times X \to E$ satisfying

i) $d(x, y) \geq \theta$, $\forall\, x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$.

ii) $d(x, y) = d(y, x)$, $\forall\, x, y \in X$.

iii) $d(x, y) \leq d(x, z) + d(z, y)$, $\forall\, x, y, z \in X$.

Then d is called the cone metric and the pair (X, d) is called the cone metric space.

E.g. 1 [5]: Let $E = R^2$ and $P = \{(x, y) \in R^2 : x \geq 0, y \geq 0\}$, $X = R$ and $d(x, y) = (|x - y|, \alpha |x - y|), \ \forall x, y \in X, \alpha \geq 0$. Then (X, d) is a cone metric space and P is a normal cone with normal constant 1.

There are two different kinds of cones: normal (with a normal constant K) and non-normal cones. Let E be a real Banach space, $P \subset E$ a cone and $\leq$ the partial ordering defined by P.
Then P is said to be normal if there exist positive real number K such that for all $x, y \in P$

$$\theta \leq x \leq y \text{ implies } \|x\| \leq K \|y\|.$$

Or, equivalently if $x_n \leq y_n \leq z_n, \ \forall n$ and $\lim_{n \to \infty} x_n = \lim_{n \to \infty} z_n = x \Rightarrow \lim_{n \to \infty} y_n = x$.

The least of all such constant K is known as normal constant.

Definition 1.3 [7]: Let X be a set and $d : X \times X \to E$ satisfying

i) $d(x, y) \geq \theta, \ \forall x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$.

ii) $d(x, y) = d(y, x), \ \forall x, y \in X$.

iii) $d(x, y) \leq d(x, z) + d(z, w) + d(w, y), \ \forall x, y, z, w \in X \ \ and \ \ w, z \in X \setminus \{x, y\}$.

Then d is called the cone rectangular metric and the pair (X, d) is called the cone rectangular metric space. Every cone metric space is a cone rectangular metric space but the converse is not true in general.

Example 2 [7]: Let $X = N$ and $E = R^2$ and $P = \{(x, y) \in E : x, y \geq 0\}$. define $d : X \times X \to E$ as follows:

$$d(x, y) = \begin{cases} (0,0), & if \ \ x = y \\ (3,9), & if \ \ x \ and \ \ y \ are \ \ in \ \{1,2\} \ but \ \ x \neq y \\ (1,3), & otherwise \end{cases}$$

Now (X, d) is a cone rectangular metric space but not cone metric space because it lacks triangular property:
$(3,9) = d(1,2) > d(1,3) + d(3,2) = (1,3) + (1,3) = (2,6)$ as $(3,9) - (2,6) = (1,3) \in P$.

Example 3 [10]: Let $X = \{a, b, c, e\}$ and $E = R^2$ and $P = \{(x, y) \in E : x, y \geq 0\}$. define

$d : X \times X \to E$ as follows:

$$\begin{cases} d(x, x) = (0,0), \ \ \forall x \in X, \\ d(y, x) = d(x, y), \forall x, y \in X, \\ d(a, b) = (3, \alpha), \\ d(a, c) = d(b, c) = (1, \alpha), \\ d(a, e) = d(b, e) = d(c, e) = (2, \alpha), \end{cases}$$

Where $\alpha > 0$ is a constant. Then (X, d) is a cone rectangular metric space but not cone metric space since it lacks triangular property:
$(3, \alpha) = d(a, b)$ and $d(a, c) + d(c, b) = (1, \alpha) + (1, \alpha) = (2, 2\alpha)$ but
$d(a, c) + d(c, b) - d(a, b) = (2, 2\alpha) - (3, \alpha) = (-1, \alpha) \notin P$ therefore $d(a, b) \leq d(a, c) + d(c, b)$ is not true.

Lemma 2.1[17]: Let P a cone and $x, y, z \in E$. . We have the following properties

i) $P + \text{int } P \subseteq \text{int } P$. in particular we have $\text{int } P + \text{int } P \subseteq \text{int } P$.

ii) $\lambda \text{ int } P \subseteq \text{int } P$. where $\lambda > 0$.

Remark 2.2[19]: Let P a cone and $x, y, z \in E$.

i) If $x \ll y$ and $y \ll z$ then $x \ll z$.

ii) If $x \ll y$ and $y \le z$ then $x \ll z$.

Definition 2.3 [7]: Let (X, d) be a cone rectangular metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $c \gg \theta$ there is N such that for all $n > N$, $d(x_n, x) \ll c$. Then $\{x_n\}$ is said to be convergent to x and x is the limit of $\{x_n\}$. We denote this by

$$\lim_{n\to\infty} x_n = x, \text{ or } x_n \to x (n \to \infty).$$

Definition 2.4 [7]: Let (X, d) be a cone rectangular metric space. Let $\{x_n\}$ be a sequence in X. If for any $c \in E$ with $c \gg \theta$ there is N such that for all $n > N$, $d(x_n, x_{n+m}) \ll c$. Then $\{x_n\}$ is said to be a Cauchy sequence in X.

Lemma 2.5[7]: Let (X, d) be a cone rectangular metric space and P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $\|d(x, x_n)\| \to 0$ as $n \to \infty$.

Lemma 2.6[7]: Let (X, d) be a cone rectangular metric space and P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $\|d(x_n, x_{n+m})\| \to 0$ as $n \to \infty$.

Lemma 2.7[5]: Let (X, d) be a cone metric space and P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y then $x = y$.

Lemma 2.8 [5]: Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ Converges to x and $\{x_n\}$ is a Cauchy sequence.

But the above two lemma's, namely lemma 2.7 and lemma 2.8 is not true in general in case of cone rectangular metric space. Consider the following example

Example 4[10]: Let $E = R$ and $P = \{x \in R : x \ge 0\}$. let $\{x_n\}$ be a sequence in Q and $a, b \in R \setminus Q, a \ne b$. We put $X = \{x_1, x_2, \ldots\ldots, x_n, \ldots\ldots\} \cup \{a, b\}$ and we consider $d : X \times X \to E$ as follows:

$$\begin{cases} d(x,x) = (0,0), \ \forall x \in X, \\ d(y,x) = d(x,y), \forall x, y \in X, \\ d(x_n, x_m) = 1, \forall n, m \in N, n \ne m, \\ d(x_n, b) = \dfrac{1}{n}, \forall n \in N, \\ d(x_n, a) = \dfrac{1}{n}, \forall n \in N, \\ d(a,b) = 1. \end{cases}$$

We see that (X, d) is not a cone metric space because we have

$$d(x_2, x_3) > d(x_2, b) + d(b, x_3) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

However, (X, d) is a cone rectangular metric space. Now since $d(x_n, a) = \frac{1}{n} \to 0,\ as\ n \to \infty$ we obtain that $x_n \to a,\ as\ n \to \infty$. Also $d(x_n, b) = \frac{1}{n} \to 0,\ as\ n \to \infty$ then $x_n \to b,\ as\ n \to \infty$.

Although the sequence $\{x_n\}$ is convergent but we have $d(x_n, x_m) = 1$ which means that $\{x_n\}$ is not Cauchy.

Definition 2.9[7]: If every Cauchy sequence is convergent in (X, d), then (X, d) is called a complete cone rectangular metric space.

Definition 2.10 [11]: Let f and g be two self maps on X. If $fw = gw = v$ for some $w \in X$ then w is the coincidence point of f and g and v is the point of coincidence of f and g.

Definition 2.11[11]: Let (X, d) be a cone metric space. If every Cauchy sequence is convergent then X is said to be complete cone metric space

Let (X, d) be a cone metric space.

i) If $u \leq v$ and $v << w$ then $u << w$.

ii) If $\theta \leq u << c$ for each $c \in \operatorname{int} P$, then $u = \theta$.

iii) If E is a real Banach space with cone P and if $a \leq a\lambda$ where $0 \leq \lambda < 1$ and $a \in P$, then $a = \theta$.

Definition 2.12 [4]: Let S and T be two self-mappings of a cone metric space (X, d). The pair (S, T) is said to be weakly compatible if $STu = TSu$ whenever $Su = Tu$ for some $u \in X$.

The following proposition is proved in [11].

Proposition 2.13 [11]: If f and g be two weakly compatible maps on X. If f and g have unique point of coincidence $fw = gw = v$ then v is the unique common fixed point of f and g.

Definition2.14 [8]:

Let be a cone and Φ be the set of all continuous non decreasing function $\varphi : P \to P$ such that

i) $\theta < \varphi(t) < t$ for all $t \in P \setminus \{\theta\}$.

ii) the series $\sum_{n \geq 0} \varphi^n(t)$ converges for all $t \in P \setminus \{\theta\}$.

Theorem 2.15[8]: Let (X, d) be complete cone rectangular metric space let $T : X \to X$ satisfy the following:

$$d(Tx, Ty) \leq \varphi(d(x, y)),\ \forall\ x, y \in X, \quad \text{Where } \varphi \in \Phi$$

Then T has a unique fixed point in X.

The following result have been proved by M. Jleli, and B. Samet [10]

Theorem 2.16[10]: Let (X, d) be cone rectangular metric space and P be a normal cone with normal constant K. Suppose that a mapping $T : X \to X$ satisfies the contractive conditions

$$d(Tx, Ty) \leq \alpha(d(Tx, x) + d(Ty, y)), \forall\ x, y \in X,$$

for all $x, y \in X$ Where $\alpha \in [0, \tfrac{1}{2})$. Then

i) T has a unique fixed point in X.

ii) For any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Theorem 2.17 [14]: Let (X, d) be cone rectangular metric space and suppose $f, g, h : X \to X$ be three functions satisfying

$$d(fx, gy) \leq \lambda d(hx, hy),\ \forall\ x, y \in X,$$

Where $\lambda \in [0,1)$. . If $f(X) \cup g(X) \subseteq h(X)$ and $h(X)$ is a complete subspace of X. Then f, g and h have a unique point of coincidence. Moreover, if (f, h) and (g, h) are weakly compatible, then

f, g and h have a unique common fixed point.

Main result

First we present few results whose proofs are similar to D.Türkoglu, M. Abuloha [19, Lemma 2, Lemma 3, Proposition 1].

Lemma 3.1: Let (X, d) be Cone rectangular metric space. Then for each $c >> 0, c \in E$, there exist $\delta > 0$ such that $(c - x) \in \operatorname{int} P$ whenever $\|x\| < \delta, x \in E$.

Lemma 3.2: Let (X, d) be Cone rectangular metric space. Then for each $c_1 >> 0$ and $c_2 >> 0$, $c_1, c_2 \in E$, there exist $c >> 0, c \in E$ such that $c << c_1$ and $c << c_2$.

For $c >> 0, c \in E$, let $B(x, c) = \{y \in X \mid d(x, y) << c\}$ and $\beta = \{B(x, c) \mid x \in X, 0 << c\}$. Then, the collection $\tau_c = \{U \subset X : \forall x \in X, \exists B \in \beta, x \in B \subset U\}$ is a topology on X.

Proposition 3.3: Every cone rectangular metric space is a topological space.

But here we show that Cone rectangular metric space need not be a Hausdorff space. Consider the following example.

Let $E = R$ and $P = \{x \in R : x \geq 0\}$. let $\{x_n\}$ be a sequence in Q and $a, b \in R \setminus Q, a \neq b$. we put $X = \{x_1, x_2, ..., x_n, ...\} \cup \{a, b\}$ and we consider $d : X \times X \to E$ as follows:

$$\begin{cases} d(x, x) = (0,0), \ \forall x \in X, \\ d(y, x) = d(x, y), \forall x, y \in X, \\ d(x_n, x_m) = 1, \forall n, m \in N, n \neq m, \\ d(x_n, b) = \dfrac{1}{n}, \forall n \in N, \\ d(x_n, a) = \dfrac{1}{n}, \forall n \in N, \\ d(a, b) = 1. \end{cases},$$

Then $\operatorname{int} P = \{x \in R \mid x > 0\}$. Now $a \neq b, a, b \in X$. also we have $d(a, b) = 1$.

Consider $B(a, \frac{1}{3}) = \{x \in X \mid d(a, x) << \frac{1}{3}\}$ and $B(b, \frac{1}{3}) = \{x \in X \mid d(b, x) << \frac{1}{3}\}$.

Now consider $x_4 \in X$ then $d(a, x_4) = \frac{1}{4}$ and $\frac{1}{3} - \frac{1}{4} = \frac{1}{12} > 0$.

Therefore $\frac{1}{3} - \frac{1}{4} \in \operatorname{int} P$ i.e.,

$$\frac{1}{3} - d(a, x_4) \in \operatorname{int} P \Rightarrow d(a, x_4) << \frac{1}{3} \Rightarrow x_4 \in B(a, \frac{1}{3}).$$

Again $B(b, \frac{1}{3}) = \{x \in X \mid d(a, x) << \frac{1}{3}\}$

We obtain that $x_4 \in B(b, \frac{1}{3})$. which implies that $B(a, \frac{1}{3}) \cap B(b, \frac{1}{3}) \neq \phi$.

Theorem 3.4: Let (X, d) be cone rectangular metric space. Let $f, g, h : X \to X$ be three functions satisfying

$$d(fx, gy) \leq \varphi(d(hx, hy)), \ \forall \ x, y \in X,$$

Where $\varphi \in \Phi$. If $f(X) \cup g(X) \subseteq h(X)$ and $h(X)$ is a complete subspace of X. Then f, g and h

have a unique point of coincidence. Moreover, if (f,h) and (g,h) are weakly compatible, then f, g and h have a unique common fixed point.

Proof: Let $x_0 \in X$ be an arbitrary point of X. Since $f(X) \subset h(X)$ then there exist $x_1 \in X$ such that $f(x_0) = h(x_1)$. Again for $x_1 \in X$ and since $g(X) \subset h(X)$ there exist $x_2 \in X$ such that $g(x_1) = h(x_2)$ and so on. Therefore, we have $h(x_{n+1}) = f(x_n)$ and $h(x_{n+2}) = g(x_{n+1})$ $n = 0,1,2,3,...$

Then

$$\begin{aligned} d(hx_n, hx_{n+1}) &= d(fx_{n-1}, gx_n) \\ &\leq \varphi(d(hx_{n-1}, hx_n)) \\ &= \varphi(d(fx_{n-2}, gx_{n-1})) \\ &\leq \varphi^2(d(hx_{n-2}, hx_{n-1})) \\ &\leq .... \\ &\leq \varphi^n(d(hx_0, hx_1)) \end{aligned}$$

Similarly for $k = 0,1,2,3,...$ we get,

$$\begin{aligned} d(hx_n, hx_{n+2k}) &\leq \varphi^n(d(hx_0, hx_{2k})) \\ d(hx_n, hx_{n+2k+1}) &\leq \varphi^n(d(hx_0, hx_{2k+1})) \end{aligned} \tag{2}$$

By using rectangular property and (2) we get,

$$\begin{aligned} d(hx_0, hx_4) &\leq d(hx_0, hx_1) + d(hx_1, hx_2) + d(hx_2, hx_4) \\ &\leq d(hx_0, hx_1) + \varphi(d(hx_0, hx_1)) + \varphi^2(d(hx_0, hx_2)) \\ d(hx_0, hx_6) &\leq d(hx_0, hx_1) + d(hx_1, hx_2) + d(hx_2, hx_3) + d(hx_3, hx_4) + d(hx_4, hx_6) \\ &\leq d(hx_0, hx_1) + \varphi(d(hx_0, hx_1)) + \varphi^2(d(hx_0, hx_1)) + \varphi^3(d(hx_0, hx_1)) + \varphi^4(d(hx_0, hx_2)) \\ &\leq \sum_{i=0}^{3} \varphi^i(d(hx_0, hx_1)) + \varphi^4(d(hx_0, hx_2)). \end{aligned}$$

By induction we have for each $k = 2,3,4,........$

$$d(hx_0, hx_{2k}) \leq \sum_{i=0}^{2k-3} \varphi^i(d(hx_0, hx_1)) + \varphi^{2k-2}(d(hx_0, hx_2)). \tag{3}$$

Also by using rectangular inequality and (2) we get,

$$\begin{aligned} d(hx_0, hx_5) &\leq d(hx_0, hx_1) + d(hx_1, hx_2) + d(hx_2, hx_3) + d(hx_3, hx_4) + d(hx_4, hx_5) \\ &\leq \sum_{i=0}^{4} \varphi^i(d(hx_0, hx_1)). \end{aligned}$$

By induction we have for each $k = 0,1,2,3,4,........$

$$d(hx_0, hx_{2k+1}) \leq \sum_{i=0}^{2k} \varphi^i(d(hx_0, hx_1)). \tag{4}$$

Using (3) in (2) we get,

$$d(hx_n, hx_{n+2k}) \leq \varphi^n(d(hx_0, hx_{2k}))$$

$$\leq \varphi^n(\sum_{i=0}^{2k-3} \varphi^i(d(hx_0, hx_1)) + \varphi^{2k-2}(d(hx_0, hx_2)).$$

$$\leq \varphi^n(\sum_{i=0}^{2k-3} \varphi^i(d(hx_0, hx_1) + d(hx_0, hx_2)) + \varphi^{2k-2}(d(hx_0, hx_1) + d(hx_0, hx_2)).$$

$$\leq \varphi^n(\sum_{i=0}^{2k-2} \varphi^i(d(hx_0, hx_1) + d(hx_0, hx_2)).$$

$$\leq \varphi^n(\sum_{i=0}^{\infty} \varphi^i(d(hx_0, hx_1) + d(hx_0, hx_2))$$

Using (4) in (2) we get,

$$d(hx_n, hx_{n+2k+1}) \leq \varphi^n(d(hx_0, hx_{2k+1}))$$

$$\leq \varphi^n(\sum_{i=0}^{2k} \varphi^i(d(hx_0, hx_1))$$

$$\leq \varphi^n(\sum_{i=0}^{2k} \varphi^i(d(hx_0, hx_1) + d(hx_0, hx_2)).$$

$$\leq \varphi^n(\sum_{i=0}^{\infty} \varphi^i(d(hx_0, hx_1) + d(hx_0, hx_2)).$$

Hence for each m we conclude

$$d(hx_n, hx_{n+m}) \leq \varphi^n(\sum_{i=0}^{\infty} \varphi^i(d(hx_0, hx_1) + d(hx_0, hx_2)) \ . \qquad (4)$$

Since $\sum_{i=0}^{\infty} \varphi^i(d(hx_0, hx_1) + d(hx_0, hx_2))$ converges, where $d(hx_0, hx_1) + d(hx_0, hx_2) \in P \setminus \{\theta\}$ and P is closed, then $\sum_{i=0}^{\infty} \varphi^i(d(hx_0, hx_1) + d(hx_0, hx_2)) \in P \setminus \{\theta\}$. hence

$$\varphi^n(\sum_{i=0}^{\infty} \varphi^i(d(hx_0, hx_1) + d(hx_0, hx_2)) \to \theta \text{ as } n \to \infty .$$

Hence given $c >> \theta$ there exist a natural number $n_0 \in N$ such that

$$\varphi^n(\sum_{i=0}^{\infty} \varphi^i(d(hx_0, hx_1) + d(hx_0, hx_2)) << c, \ \forall n \geq n_0 . \qquad (5)$$

Hence from (4) and (5), we get,

$$d(hx_n, hx_{n+m}) << c, \ \forall n \geq n_0.$$

This implies that $\{hx_n\}$ is a Cauchy sequence in X . Since $h(X)$ is a complete subspace of X , there exist a point $v \in X$ such that $hx_n \to v$. Also, we can have a point $u \in X$ such that $h(u) = v$

We will prove $h(u) = f(u)$.

Given $c >> \theta$, we choose natural numbers k_1, k_2 such that

$d(hx_n, v) \ll \frac{c}{3} \quad \forall n \geq k_1, \; d(hx_n, hx_{n+1}) \ll \frac{c}{3} \quad \forall n \geq k_2.$

By rectangular property we have,

$$\begin{aligned} d(hu, fu) &\leq d(hu, hx_n) + d(hx_n, hx_{n+1}) + d(hx_{n+1}, fu) \\ &\leq d(v, hx_n) + d(hx_n, hx_{n+1}) + d(gx_n, fu) \\ &\leq d(v, hx_n) + d(hx_n, hx_{n+1}) + \varphi(d(hx_n, hu)) \\ &\leq d(v, hx_n) + d(hx_n, hx_{n+1}) + d(hx_n, hu)) \\ &\ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c. \quad \forall n \geq k. \end{aligned}$$

Where $k = \max\{k_1, k_2\}$.

Since c is arbitrary we have $d(hu, fu) = \theta$ therefore we get $h(u) = f(u) = v$.

Similarly, we can prove that $h(u) = g(u)$ so that we get $h(u) = f(u) = g(u) = v$. Which implies that v point of coincidence of f, g and h. Now we show that the point of coincidence is unique, if not let v^* be another point of coincidence of f, g and h. that is $h(u^*) = f(u^*) = g(u^*) = v^*$ for some $u^* \in X$.

Then,

$$\begin{aligned} d(v, v^*) &= d(fu, gu^*). \\ &\leq \varphi(d(hu, hu^*)). \\ &\leq \varphi(d(v, v^*)). \end{aligned}$$

This implies that $v^* = v$. which proves the uniqueness of point of coincidence. Now since (f, h) and (g, h) are weakly compatible, then by the proposition 2.1 we conclude that f, g and h have a unique common fixed point.

The following corollaries are the main result in [8]

Corollary 3.5: Let (X, d) be cone rectangular metric space. Let $T: X \to X$ satisfying

$$d(Tx, Ty) \leq \varphi(d(x, y)), \quad \forall \; x, y \in X,$$

Where $\varphi \in \Phi$. Then T has a unique fixed point in X.

Proof: Consider $f(u) = g(u) = T(u)$ and $h(u) =$ identity mapping in the theorem 3.4, we get the desired result.

Corollary 3.6: Let (X, d) be cone rectangular metric space. Let $f, h: X \to X$ be three functions satisfying

$$d(fx, fy) \leq \varphi(d(hx, hy)), \quad \forall \; x, y \in X,$$

Where $\varphi \in \Phi$. If $f(X) \subseteq h(X)$ and $h(X)$ is a complete subspace of X. Then f and h have a unique point of coincidence. Moreover, if (f, h) is weakly compatible, then f and h have a unique common fixed point in X.

Proof: Consider $f(u) = g(u)$ in the theorem 3.4, we get the desired result.

The following corollary is the main result in [14]

Corollary 3.7: Let (X, d) be cone rectangular metric space. Let $f, g, h: X \to X$ be three functions satisfying

$$d(fx, gy) \leq \lambda(d(hx, hy)), \quad \forall \; x, y \in X,$$

Where $\lambda \in [0,1)$. If $f(X) \cup g(X) \subseteq h(X)$ and $h(X)$ is a complete subspace of X. Then f, g and h have a unique point of coincidence. Moreover, if (f, h) and (g, h) are weakly compatible, then f, g and h have a unique common fixed point.

Proof: Consider $\varphi(t) = \lambda t$ in the theorem 3.4, we get the desired result

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