Generalisations of Eneströ M-Kakeya Theorem for Complex Polynomial

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ABSTRACT:

Let $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ *be a polynomial of degree* n *. In this paper we have obtained some generalizations of Eneströ m -Kakeya Theorem regarding the region for location of zeros of polynomial in terms of restricted coefficients. Our results give not only zero free regions for polynomials but also the number of zeros that can lie in a prescribed region. Our result sharpens as well as generalizes the earlier known results.*

Key-Words: - *Polynomials; Zeros; Inequalities; Complex domain.*

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1-INTRODUCTION AND STATEMENT OF RESULTS

The following results are well known in the theory of the distribution of zeros of polynomial.

THEOREM A: - If $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n with the coefficients satisfying the condition

$$a_n \geq a_{n-1} \geq a_{n-2} \geq \cdots\cdots \geq a_1 \geq a_0 > 0 \ , \qquad (1)$$

then all zeros of $p(z)$ lie in

$$|z| \leq 1 \, . \qquad (2)$$

This is known as Eneströ m-Kakeya theorem [1, 2].

Joyal, Labelle and Rahman [4] extended Theorem A to the polynomials with coefficients not necessarily non-negative. More precisely, they proved the following

THEOREM B. *Let* $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree n, such that*

$$a_n \geq a_{n-1} \geq \cdots\cdots \geq a_1 \geq a_0 \ . \qquad (3)$$

Then $p(z)$ *has all its zeros in*

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|} \ . \qquad (4)$$

If $a_0 > 0$, then this result reduces to Theorem A.

THEOREM C: - If $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n . If $a_j=\alpha_j+i\beta_j$ and $\mathrm{Re}(a_k)=\alpha_k$, $\mathrm{Im}(a_k)=\beta_k$ for $k=0,1,2,\ldots\ldots,n$ and

$$\alpha_n \ge \alpha_{n-1} \ge \alpha_{n-2} \ge \cdots\cdots \ge \alpha_1 \ge \alpha_0 \ge 0, \tag{5}$$

then $p(z)$ has all its zeros in the ring-shaped region given by

$$\frac{|a_0|}{R_1^{\,n-1}[2R_1\alpha_n + R_1|\beta_n| - (\alpha_0+|\beta_0|)]} \le |z| \le R_1 = 1+\frac{1}{\alpha_n}\left[2\sum_{k=0}^{n-1}|\beta_k| + |\beta_n|\right]. \tag{6}$$

The above result is due to Govil and Rahman [3].

Govil and Rahman [3] generalized Eneström-Kakeya Theorem for polynomials with complex coefficients by considering the moduli of the coefficients to be monotonically increasing and proved the following

THEOREM D. *Let* $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree n, with complex coefficients such that for some real* β

$$|\arg a_\nu - \beta| \le \alpha \le \frac{\pi}{2} \qquad \nu = 0,1,2,\ldots\ldots,n,$$

and

$$|a_n| \ge |a_{n-1}| \ge \cdots\cdots \ge |a_1| \ge |a_0|. \tag{7}$$

Then $p(z)$ *has all its zeros in*

$$|z| \le (\cos\alpha + \sin\alpha) + 2\frac{\sin\alpha}{|a_n|}\sum_{j=0}^{n-1}|a_j|. \tag{8}$$

In this paper firstly we prove the following result for polynomials with complex coefficients, which improves upon Theorem D in particular case and also improves upon other results. More precisely, we prove

THEOREM 1. *If* $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ *be a polynomial of degree n, with complex coefficients such that for some real* β ,

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2} \quad ,j=0,1,2\ldots\ldots n$$

and for some $K \ge 1$,

$$K|a_n| \ge |a_{n-1}| \ge \cdots\cdots \ge |a_\lambda| \le |a_{\lambda-1}| \le \cdots\cdots \le |a_1| \le |a_0|, \tag{9}$$

then p(z) *has no zero in* $|z| < \frac{|a_0|}{M_1}$ *and the maximum number of zeros in* $\frac{|a_0|}{M_1} < |z| \le \delta,\ 0<\delta<1$ *does not exceed*

$$\frac{1}{\log(1/\delta)}\log\left[\frac{\left(K|a_n|+|a_0|\right)(1+\cos\alpha+\sin\alpha)-2|a_\lambda|\cos\alpha+2\sin\alpha\sum_{j=1}^{n-1}|a_j|}{|a_0|}\right],$$

where

$$M_1 = K|a_n|(1+\cos\alpha+\sin\alpha)+|a_0|(\cos\alpha+\sin\alpha)-2|a_\lambda|\cos\alpha + 2\sin\alpha\sum_{j=1}^{n-1}|a_j|. \quad (10)$$

We now find a disk containing all zeros of $p(z)$ under the hypothesis of Theorem 1, improving the bound of Theorem D for K=1 and $\lambda = 0$, as following

THEOREM 2. *If* $p(z)=\sum_{v=0}^{n} a_v z^v$ *be a polynomial of degree n, with complex coefficients such that for some real* β ,

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}, \quad j=0,1,2\ldots\ldots n,$$

and for some $K \ge 1$,

$$K|a_n| \ge |a_{n-1}| \ge \cdots\cdots \ge |a_\lambda| \le |a_{\lambda-1}| \le \cdots\cdots \le |a_1| \le |a_0|, \quad (11)$$

then all the zeros of $p(z)$ *lie in*

$$|z+K-1| \le \frac{1}{|a_n|}\left\{ K|a_n|(\cos\alpha+\sin\alpha)+|a_0|(1+\cos\alpha+\sin\alpha) - 2|a_\lambda|\cos\alpha + 2\sin\alpha\sum_{j=1}^{n-1}|a_j| \right\}. \quad (12)$$

REMARK 3. For the case $\lambda = 0$, Theorem 2 reduces to a result due to Shah and Liman [5, Theorem 1]. For the case K=1 and $\lambda = 0$, Theorem 2 gives the following

COROLLARY 4. *If* $p(z)=\sum_{v=0}^{n} a_v z^v$ *be a polynomial of degree n, with complex coefficients such that for some real* β ,

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}, \quad j=0,1,2\ldots\ldots n.,$$

$$|a_n| \ge |a_{n-1}| \ge \cdots\cdots \ge |a_0|, \quad (13)$$

then all the zeros of $p(z)$ *lie in*

$$|z| \leq \frac{1}{|a_n|}\left\{|a_n|(\cos\alpha + \sin\alpha) + |a_0|(1 - \cos\alpha - \sin\alpha) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|\right\}. \tag{14}$$

Corollary 4 always gives better bound than Theorem D due to Govil and Rahman [3], except in the case, $|a_0| = 0$ or $\alpha = 0$ or $(\pi/2)$.

2. LEMMAS

We need the following lemma for the proof of the above theorems.

LEMMA 2.1. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree n, such that for some real* β ,

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad j = 0, 1, 2, \ldots\ldots, n,$$

then for some $t > 0$,

$$|ta_j - a_{j-1}| \leq \big|t|a_j| - |a_{j-1}|\big|\cos\alpha + (t|a_j| + |a_{j-1}|)\sin\alpha. \tag{15}$$

The above lemma follows from inequality (6) in [3].

3. PROOF OF THE THEOREMS

PROOF OF THEOREM 1. Consider the polynomial

$$F(z) = (1 - z)p(z)$$

$$= (1 - z)(a_0 + a_1 z + a_2 z^2 + \cdots\cdots + a_n z^n)$$

$$= a_0 - a_n z^{n+1} + \sum_{j=1}^{n} (a_j - a_{j-1}) z^j$$

$$= a_0 - a_n z^{n+1} + (a_n - a_{n-1}) z^n + \sum_{j=1}^{n-1} (a_j - a_{j-1}) z^j$$

$$= a_0 - a_n z^{n+1} + (Ka_n - a_{n-1} + a_n - Ka_n) z^n + \sum_{j=1}^{n-1} (a_j - a_{j-1}) z^j.$$

For $|z| = 1$, we have

$$|F(z)| \leq |a_0| + |a_n| + |Ka_n - a_{n-1}| + (K-1)|a_n| + \sum_{j=1}^{n-1} |a_j - a_{j-1}|,$$

which on applying Lemma 2.1 for t=1, gives

$$|F(z)| \leq |a_0| + |a_n| + \big(K|a_n| - |a_{n-1}|\big)\cos\alpha + \big(K|a_n| + |a_{n-1}|\big)\sin\alpha$$
$$+ (K-1)|a_n| + \sum_{j=1}^{n-1} \big||a_j| - |a_{j-1}|\big|\cos\alpha + \sum_{j=1}^{n-1} \big(|a_j| + |a_{j-1}|\big)\sin\alpha$$

$$\leq K|a_n|\{1+\cos\alpha+\sin\alpha\}+|a_0|-|a_{n-1}|(\cos\alpha-\sin\alpha)$$
$$+\left\{\sum_{j=1}^{\lambda}|a_j-a_{j-1}|+\sum_{j=\lambda+1}^{n-1}|a_j-a_{j-1}|\right\}\cos\alpha+\left\{\sum_{j=1}^{n-1}|a_j|+\sum_{j=1}^{n-1}|a_{j-1}|\right\}\sin\alpha$$

$$=K|a_n|\{1+\cos\alpha+\sin\alpha\}+|a_0|-|a_{n-1}|(\cos\alpha-\sin\alpha)$$
$$+\left(|a_0|+|a_{n-1}|-2|a_\lambda|\right)\cos\alpha+\left\{|a_0|-|a_{n-1}|+2\sum_{j=1}^{n-1}|a_j|\right\}\sin\alpha$$
$$=\left\{K|a_n|+|a_0|\right\}(1+\cos\alpha+\sin\alpha)-2|a_\lambda|\cos\alpha+2\sin\alpha\sum_{j=1}^{n-1}|a_j|$$
$$=M\ .\quad \text{say}$$

Thus

$$|F(z)|\leq M \qquad \text{for} \qquad |z|=1\ .$$

Now, it is known (see [6], p. 171) that if $f(z)$ is regular, $f(0)\neq 0$ and $|F(z)|\leq M$ in $|z|\leq 1$; then the maximum number of zeros of $f(z)$ in $|z|\leq\delta<1$ can not exceed $\frac{1}{\log(1/\delta)}\log\left(\frac{M}{|f(0)|}\right)$. Applying this result to $F(z)$, we get the maximum number of zeros of $F(z)$ and hence of $p(z)$ in $|z|\leq\delta<1$ can not exceed

$$\frac{1}{\log(1/\delta)}\log\left[\frac{(K|a_n|+|a_0|)(1+\cos\alpha+\sin\alpha)-2|a_\lambda|\cos\alpha+2\sin\alpha\sum_{j=1}^{n-1}|a_j|}{|a_0|}\right].$$

This gives one part of Theorem 1.

Now we shall show that no zero of $p(z)$ lies in

$$|z|<\frac{|a_0|}{M_1}\ .$$

For this, we have

$$F(z)=(1-z)\,p(z)$$
$$=a_0+\sum_{j=1}^{n}(a_j-a_{j-1})z^j-a_nz^{n+1}$$
$$F(z)=a_0+h(z),\qquad \text{say} \tag{16}$$

where

$$h(z)=-a_nz^{n+1}+\sum_{j=1}^{n}(a_j-a_{j-1})z^j$$
$$=-a_nz^{n+1}+[Ka_n-a_{n-1}+a_n-Ka_n]z^n+\sum_{j=1}^{n-1}(a_j-a_{j-1})z^j\ .$$

For $|z|=1$, we have

$$\max_{|z|=1} h(z) = |a_n| + |Ka_n - a_{n-1}| + (K-1)|a_n| + \sum_{j=1}^{n-1} |a_j - a_{j-1}|$$

$$\leq |a_n| + (K-1)|a_n| + (K|a_n| - |a_{n-1}|)\cos\alpha + (K|a_n| + |a_{n-1}|)\sin\alpha$$

$$+ (|a_0| + |a_{n-1}| - 2|a_\lambda|)\cos\alpha + \left(|a_0| - |a_{n-1}| + 2\sum_{j=1}^{n-1}|a_j|\right)\sin\alpha$$

$$= K|a_n|(1+\cos\alpha+\sin\alpha) + |a_0|(\cos\alpha+\sin\alpha) - 2|a_\lambda|\cos\alpha + 2\sin\alpha\sum_{j=1}^{n-1}|a_j|$$

$= M_1$. say

Thus for $|z| = 1$, $|h(z)| \leq M_1$.

Further, since $h(0)=0$ and $h(z)$ is analytic, applying Schwarz's lemma to $h(z)$, we get

$$|h(z)| \leq M_1|z|, \quad \text{for} \quad |z| \leq 1.$$

Also

$$F(z) = a_0 + h(z)$$

for $|z| \leq 1$, equation (16) gives

$$|F(z)| \geq |a_0| - |h(z)|$$

$$\geq |a_0| - M_1|z|$$

$$> 0,$$

if

$$|z| < \frac{|a_0|}{M_1}.$$

Hence $F(z)$ and therefore $p(z)$ has no zero in

$$|z| < \frac{|a_0|}{M_1}.$$

This proves Theorem 1 completely.

PROOF OF THEOREM 2. Consider the polynomial

$$F(z) = (1-z)p(z)$$

$$= (1-z)(a_0 + a_1 z + a_2 z^2 + \cdots\cdots + a_n z^n)$$

$$F(z) = -a_n z^{n+1} + (a_n - a_{n-1})z^n + \cdots\cdots + (a_1 - a_0)z + a_0.$$

Then for $|z| > 1$, we have

$$|F(z)| = \left| -a_n z^{n+1} + (a_n - a_{n-1}) z^n + \cdots\cdots + (a_1 - a_0) z + a_0 \right|$$

$$= \left| \begin{array}{c} -a_n z^{n+1} + a_n z^n - K a_n z^n + (K a_n - a_{n-1}) z^n + (a_{n-1} - a_{n-2}) z^{n-1} \\ + \cdots\cdots + (a_1 - a_0) z + a_0 \end{array} \right|$$

$$\geq |a_n||z|^n |z + K - 1| - |z|^n \left\{ \begin{array}{c} |K a_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} + \frac{|a_{n-2} - a_{n-3}|}{|z|^2} \\ + - - - - - - - - - - - - + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \end{array} \right\}.$$

On applying Lemma 2.1 and the fact $\frac{1}{|z|^j} < 1,\ j = 1, 2, \ldots\ldots n$, we get (as $|z| > 1$)

$$|F(z)| > |z|^n \left[|a_n||z + K - 1| - \left\{ \begin{array}{c} |K a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}| \\ + \ldots\ldots + |a_1 - a_0| + |a_0| \end{array} \right\} \right]$$

$$> |z|^n \left[|a_n||z + K - 1| - \left\{ \begin{array}{c} (K|a_n| - |a_{n-1}|)\cos\alpha + (K|a_n| + |a_{n-1}|)\sin\alpha \\ + \ldots\ldots\ldots + \sum_{j=1}^{n-1} |a_j - a_{j-1}| + |a_0| \end{array} \right\} \right]$$

$$= |z|^n \left[|a_n||z + K - 1| - \left\{ \begin{array}{c} (K|a_n| - |a_{n-1}|)\cos\alpha + (K|a_n| + |a_{n-1}|)\sin\alpha + |a_0| \\ + \sum_{j=1}^{\lambda} |a_j - a_{j-1}| + \sum_{j=\lambda+1}^{n-1} |a_j - a_{j-1}| \end{array} \right\} \right]$$

$$= |z|^n \left[|a_n||z + K - 1| - \left\{ \begin{array}{c} (K|a_n| - |a_{n-1}|)\cos\alpha + (K|a_n| + |a_{n-1}|)\sin\alpha + |a_0| \\ + (|a_0| + |a_{n-1}| - 2|a_\lambda|)\cos\alpha + \left(|a_0| - |a_{n-1}| + 2\sum_{j=1}^{n-1} |a_j| \right)\sin\alpha \end{array} \right\} \right]$$

$$= |z|^n \left[|a_n||z + K - 1| - \left\{ \begin{array}{c} (K|a_n| + |a_0|)\cos\alpha + (K|a_n| + |a_0|)\sin\alpha \\ + 2\sum_{j=1}^{n-1} |a_j| \sin\alpha - 2|a_\lambda|\cos\alpha + |a_0| \end{array} \right\} \right]$$

$$= |z|^n \left[|a_n||z + K - 1| - \left\{ \begin{array}{c} K|a_n|(\cos\alpha + \sin\alpha) + |a_0|(1 + \cos\alpha + \sin\alpha) \\ - 2|a_\lambda|\cos\alpha + 2\sin\alpha \sum_{j=1}^{n-1} |a_j| \end{array} \right\} \right]$$

> 0,

if

$$\left|z+K-1\right| > \frac{1}{\left|a_n\right|}\begin{bmatrix} K\left|a_n\right|(\cos\alpha+\sin\alpha)+\left|a_0\right|(1+\cos\alpha+\sin\alpha) \\ -2\left|a_\lambda\right|\cos\alpha+2\sin\alpha\sum_{j=1}^{n-1}\left|a_j\right| \end{bmatrix}.$$

Hence all the zeros of $F(z)$ with $\left|z\right| > 1$ lie in

$$\left|z+K-1\right| \leq \frac{1}{\left|a_n\right|}\begin{bmatrix} K\left|a_n\right|(\cos\alpha+\sin\alpha)+\left|a_0\right|(1+\cos\alpha+\sin\alpha) \\ -2\left|a_\lambda\right|\cos\alpha+2\sin\alpha\sum_{j=1}^{n-1}\left|a_j\right| \end{bmatrix}. \quad (17)$$

But those zeros of $F(z)$ with $\left|z\right| \leq 1$ already satisfy the inequality (17). Since all the zeros of $p(z)$ are also the zeros of $F(z)$, therefore it follows that all the zeros of $p(z)$ lie in the disk defined by inequality (17) and this completes the proof of the Theorem 2.

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