Formulation of Finite Element Method for 1-D Poisson Equation

Mrs. Pallavi P. Chopade[#], Dr. Prabha S. Rastogi[*]
[#]*Research Scholar,* [*]*Department of Mathematics*
Shri. J. J. T. University, Jhunjhunu, Rajasthan, India

Abstract -*This paper focuses on the use of solving electrostatic one-dimension Poisson differential equation boundary-value problem. Sample problems that introduce the finite element methods are presented here and evaluated with analytical and numerical approaches. These approaches are developed in MATLAB and their solutions are compared and verified. Error analysis is also presented in this paper where the numerical error is compute dusing two different definitions namely the percent error and the error based on the* L_2 *norm. this numerical error is reduced by increasing in number of elements.*

Keywords: *Boundary-value problems, Differential equation, Electrostatics, Finite Element Method, Poisson's equation.*

I. INTRODUCTION

Differential equations have wider applications in various fields of engineering and science disciplines. Generally, these are used to model variations of any physical quantity (temperature, pressure, displacement, stress, etc) with respect to time 't' or space having coordinates (x,y,z).Based on the involvement of ordinary or partial derivatives, differential equations can be classified as an ordinary differential equation or partial differential equation. They can be also classified as linear/non-linear.

The Poisson's equation, Fourier equation, heat equation and Poisson's equation are among the most prominent PDEs that undergraduate engineering students will encounter. The usual practice is to introduce the student to the analytical solution of these equations via the method of separation of variables. Under the assumption of linearity, the method naturally leads to the formulation of solutions as Fourier series expansions.

The numerical solution of differential equations has become one of powerful activity during the last sixty years or so, primarily due to advances in computer technology and the introduction of numerical computing applications like MATLAB, Mathematica, Maple which in turn has led to improvements in the numerical methods that are used. As a result, many scientific and engineering problems that involve linear and nonlinear singular ordinary differential equations and partial differential equations that were previously unsolved can now be resolved by using suitable numerical methods.

The three-dimensional Poisson equation for a function (x, y, z) , describing the electrostatic potential when unpaired electric charge is present is given as [8], [9],

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=-\frac{\rho}{\varepsilon_0} \text{ or } \nabla^2 V=-\frac{\rho}{\varepsilon_0} \#1$$

where x, y and z are the independent variables, $V$is the unknown function, and $\nabla^2$is the Laplacian operator.The solution of equation to be found is supplemented by initial and/or boundary conditions.

Consider the variation of potential in one of the three independent variables. The Poisson's equation becomes an ordinary derivative and reduces to one dimensional as[2], [3], [5],

$$\frac{d^2V}{dx^2}=-\frac{\rho}{\varepsilon_0}, \quad \text{or } \frac{d^2V}{dy^2}=-\frac{\rho}{\varepsilon_0}, \quad \text{or } \frac{d^2V}{dz^2}=-\frac{\rho}{\varepsilon_0} \#2$$

with initial and final points as boundary conditions.

In this paper finite element method is presented in the context of problems arising in electrostatics particularly one-dimensionPoisson equation with Dirichlet boundary conditions, to understand the concept of finite element method in engineering field [6].

The solution for one dimensional differential equations can be obtained using analytical methods, but it becomes more cumbersome as the dimensions increases with different boundary conditions and complex shapes. So, to ease the process of solving complicated numerical, different numerical approaches are used. In this paper, the basic concept of FEM is studied so that we can apply it to the higher dimensions with different boundary conditions. This basic concept is applied to one dimensional differential equations alongwith the algorithm developed in MATLAB.

II. FINITE ELEMENT SOLUTION TO ONE DIMENSIONAL DIFFERENTIAL EQUATION

The basic concept used in finite element method is representation of domain into smaller subdomains known as finite elements. The distribution of the primary unknown quantity inside these elements is interpolated based on the values of the edge, if vector elements are provided or based on values of nodes, if nodal elements are used. The shape function or interpolations used to distribute these elements must be a complete set of polynomials. The accuracy of the solution provided depends upon the order of the polynomials used. This solution is obtained after solving set of linear equations by one of the methods used in algebra.The linear equations are formed by converting the governing differential equation and associated boundary conditions to an integrodifferential formulation either by use of weighted residual method (Galerkin approach) or by minimizing a function[1], [4], [7].

The key steps involved in applying the Galerkin method to obtain the FEM solution of a boundary value problem are-

Step 1: Domain discretization using finite elements

Step2: Chose appropriate interpolation function or shape function or basis function

Step3: Obtain corresponding linear equation for single element by applying the boundary conditions.

Step 4: Formulate the Global matrix system of equations through assembly of all elements and then apply the Dirichlet boundary conditions.

Step5: Solve the linear equation using one of the linear algebra techniques such as Cramer's rule,etc.

Step 6: Post process the obtained results.

Here, a one-dimension electrostatic boundary value problem is studied whose solution is obtained using finite element method.The reason to choose one dimensional problem is to understand the steps involved in solving rather than dealing with extensive mathematical derivations and geometrical complications. Further, it can be extended to two-dimensional and three-dimensional problems.The accuracy and effectiveness of the FEM is evaluated by comparing the numerical result with analytical results [1], [10].

Problem definition: Consider a two infinite in extent parallel conducting plates that are positioned normal to the x-axis and separated by a distance$d$, as shown in Figure1. One plate is kept a fixed potential$V = 2$and these condplate is maintained at$V = 0$(ground). The region between the plates is filled with anon magnetic medium having a dielectric constant $\varepsilon_r$and a uniform electron volume charge density$\rho_v = -\rho_0$.Obtain the electric (orelectrostatic)potential in the region between the two parallel plates.

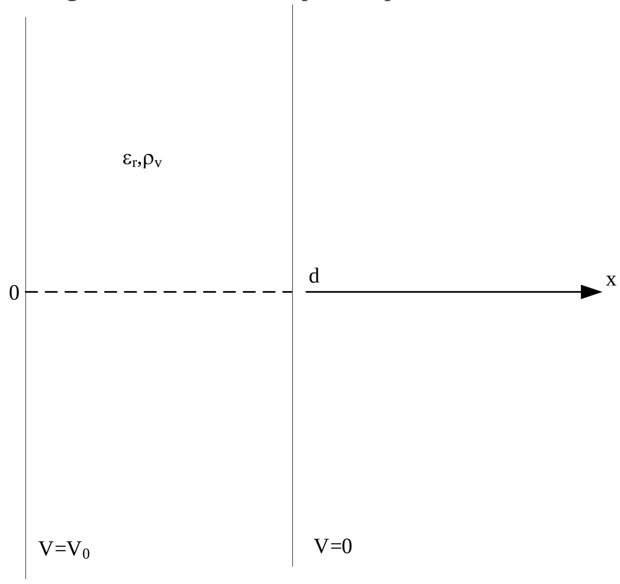

Fig. 1: Electrostatic Numerical showing boundary condition

A) Analytical Solution:

The potential distribution at any point between the two plates is governedby Poisson's equation as,

$$\nabla(\varepsilon_r \nabla V) = -\frac{\rho_v}{\varepsilon_0} \#3$$

which is subject to set of boundary conditions,

$$V(0) = V_0$$

and

$$V(d) = 0$$

For a simple nonmagnetic medium (homogeneous, linear and isotropic), Poisson's equation in one dimension can be suitably written as,

$$\frac{d^2V}{dx^2} = \frac{\rho_0}{\varepsilon_r \varepsilon_0} \#4$$

where ρ_v was replaced by $-\rho_0$. Integrating equation 4 twice, the equation of potential is,

$$V(x) = \frac{\rho_0}{2\varepsilon_r \varepsilon_0} x^2 + c_1 x + c_0 \#5$$

where c_1 and c_0 are constants to be determined from set of given Dirichletconditions. Thus, imposing the two boundary conditions in equation 5, the analytical solution takes the form,

$$V(x) = \frac{\rho_0}{2\varepsilon_r \varepsilon_0} x^2 - \left(\frac{\rho_0 d}{2\varepsilon_r \varepsilon_0} + \frac{V_0}{d} \right) x + V_0 \#6$$

B) FEM Solution

This is one of the powerful numerical method to solve the given problem. Our main objective is to compute the electric potential distribution between two parallel plates separated by a distance d and positioned normal to the x-axis.

The leftmost plate is maintained at a constant potentialV_0 whereas the rightmost plate is grounded. The region between the plates is characterizedby a dielectric constant ε_r and a uniform electron charge density $-\rho_0$.

For a finite element simulation and comparison of the numerical solution with the exact analytical solution, following parameters are considered,

$$\varepsilon_r = 1$$
$$V_0 = 2V$$
$$d = 8\ cm$$
$$\rho_0 = 10^{-8}\ C/m^3$$

Consider, the domain between plates is equally divided into four linear finite elements as shown in figure 2.

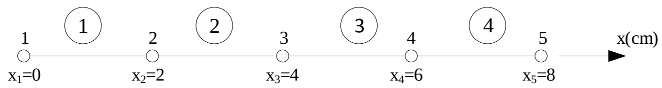

Fig. 2: Discretization of given domain in four elements

All the elements in the domain are characterized by the same length l^e and the same dielectric constant ε_r^e. Thus, the element coefficient matrix K^e is given by,

$$K^e = \frac{(8.85 * 10^{-12})}{2 * 10^{-12}} \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} = 4.425 * 10^{-10} \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \#7$$

where $\varepsilon^e = \varepsilon_r^e \varepsilon_0 = 8.85 * 10^{-12}\ F/m$ and $l^e = 2 * 10^{-2}\ m$.The element right hand side vector f^e becomes,

$$f^e = -\frac{2 * 10^{-2} * 10^{-8}}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = -10^{-10} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \#8$$

The contribution of the right-hand-side vector d^e to the global right-hand-side vector,is zero for all nodes except for the two end nodes of the domain. However, at these two end nodes, Dirichlet boundary conditions must be imposed and, therefore, the contribution by vector d^e is effectively discarded. Thus, the global matrix system for the finite elements mesh becomes,

$$4.425 * 10^{-10} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{Bmatrix} = -10^{-10} \begin{Bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 1 \end{Bmatrix} \#9$$

Evaluating further we get,

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{Bmatrix} = \begin{Bmatrix} -0.2259887 \\ -0.4519774 \\ -0.4519774 \\ -0.4519774 \\ -0.2259887 \end{Bmatrix} \#10$$

Imposing Dirichlet boundary condition, $V = 2$ at node 1, eliminates the entire first row, including the first row of the right-hand side vector and first column of coefficient matrix. Once this is done, the right-hand side vector is updated as,

$$b_i = b_i - \frac{k_i}{V_0} \#11$$

Thus, the matrix gets reduced to,

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} V_2 \\ V_3 \\ V_4 \\ V_5 \end{Bmatrix} = \begin{Bmatrix} 1.5480226 \\ -0.4519774 \\ -0.4519774 \\ -0.2259887 \end{Bmatrix} \#12$$

The secondary condition $V = 0$ at node 5 is imposed by eliminating the entire last row of matrix system and last column of coefficient matrix.

Thus, the final global matrix becomes,

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} V_2 \\ V_3 \\ V_4 \end{Bmatrix} = \begin{Bmatrix} 1.5480226 \\ -0.4519774 \\ -0.4519774 \end{Bmatrix} \#13$$

This global matric can be solved using Cramer's rule giving the values of V_2, V_3 *and* V_4i.e. the electric potential at three interior nodes of finite element domain.

$$\therefore V_2 = 0.8220339, V_3 = 0.0960451 \text{ and } V_4 = -0.1779661 \#14$$

The electric potential at intermediate points requires the use of the interpolation or shape functions employed for each finite element for plotting of graph. For the given numerical, linear interpolation functions were used and, thus, the numerical solution at intermediate points inside an element is given by,

$$V(\xi) = V_1^e N_1(\xi) + V_2^e N_2(\xi) \#15$$

where

$$N_1(\xi) = \frac{1-\xi}{2} \#16$$

$$N_2(\xi) = \frac{1+\xi}{2} \#17$$

and

$$\xi = \frac{2(x - x_1^e)}{x_2^e - x_1^e} - 1 \#18$$

Substituting these values equations 16 and 17 becomes,

$$N_1(x) = \frac{x_2^e - x}{x_2^e - x_1^e} \#19$$

and

$$N_2(x) = \frac{x - x_1^e}{x_2^e - x_1^e} \#20$$

Hence,

$$V(\xi) = V_1^e \left(\frac{x_2^e - x}{x_2^e - x_1^e}\right) + V_2^e \left(\frac{x - x_1^e}{x_2^e - x_1^e}\right) \#21$$

where V_1^e and V_2^e are the values of the electric potential at the two end nodes of the element.

C)FEM Algorithm in MATLAB

The FEM Algorithm developed in MATLAB is discussed as

- Step1: Define the parameters by initialising it.
- Step2: Discretise the domain by number of elements.
- Step 3: Calculate the element coefficient matrix.
- Step 4: Based on assembly process, calculate the global matrix system for finite element mesh by imposing the boundary conditions.
- Step 5: Post process the result.

Results

The electric potential for the four-element mesh over the complete domain is evaluated using the developed algorithm. The results obtained are shown in table 1 and is graphically plotted in figure 3.

TABLE 1: COMPARISON OF RESULTS

Sr. No.	x (Distance in meters)	Potential (V) using FEM Solution	Potential (V) using Analytical Solution
1	0	2	2
2	0.01	NaN	1.3545198

3	0.02	0.8220339	0.8030847
4	0.03	NaN	0.4025424
5	0.04	0.0960451	0.0960452
6	0.05	NaN	-0.0974576
7	0.06	-0.1779661	-0.1779661
8	0.07	NaN	-0.1454802
9	0.08	0	4.44089209e-16

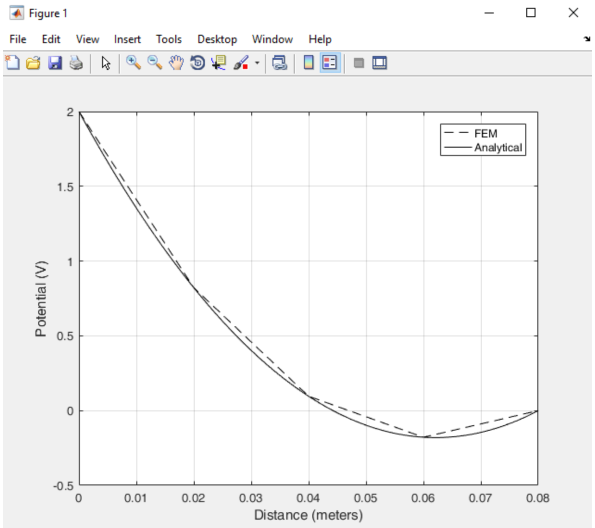

Figure 3: Comparison of solutions plotted graphically

From the Figure 3, the electric potential at the nodes of the finite element mesh matches perfectly the analytical solution, whereas at intermediate evaluation points there is a deviation between the two solutions. The reason for this deviation stems from the fact that the numerical solution at intermediate points is an interpolation of the nodal values using linear shape functions. An acceptable representation of the numerical error between the finite element solution and the analytical solution is defined as the area bounded by the two curves, which are depicted in Figure 3, as compared to the total area under the curve described by the exact solution. The numerical error is given by,

$$error(\%) = \left\{ \frac{1}{|A_{an}|} \sum_{e=1}^{N_e} \left| A_{an}^{(e)} - A_{fem}^{(e)} \right| \right\} * 100\% \#22$$

The L_2 norm that represents the distance between two methods, is computed to quantify the numerical discrepancy between the finite element solution and analytical solution is given as,

$$L_2 norm = \left\| V_{an} - V_{fem} \right\|_2 = \left\{ \sum_{e=1}^{N_e} \int_{\Omega^e} \left[V_{an}^{(e)} - V_{fem}^{(e)} \right]^2 dx \right\}^{\frac{1}{2}} \#23$$

The numerical error as function of number of linear elements is shown in Table II.

TABLE II: NUMERICAL ERROR FUNCTION

Sr. No.	Number of linear elements	Percentage error (Numerical)	Error based on L_2 norm
1	4	9.4786729	0.0116700
2	8	2.3696682	0.0029175
3	12	1.0531858	0.0012966
4	16	0.5924170	7.2937539e-04
5	20	0.3791469	4.6680025e-04
6	24	0.2632964	3.2416684e-04

From Table II it is observed that as the number of linear elements is doubling, both the error reduces at the same rate (by a factor of four). Thus, increase in the number of elements the numerical solution converges to analytical solution.

III. CONCLUSION

This paper presents understanding of Finite Element Method to solve one dimensional differential equation system for electrostatic field of engineering with Dirichlet boundary conditions. The electrical potential over the complete domain is evaluated. The FEM solution is compared with analytical solution. Both solutions give same electric potential at the nodes where as giving deviation in values at intermediate evaluation points. This deviation is because of interpolation of nodal values which uses linear shape functions. An increase in the number of elements the numerical solution approaches to analytical solution. Further, using higher order interpolation functions, the finite element solution will be accurately represented within discretized domain substantially reducing the numerical error.

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