Vertex Prime Labeling of Path Related Graphs

Mukund V. Bapat[1]

1. Abstract:

Two graphs or two copies of given graph are joined by some paths are referred as path related graphs. In this paper we discuss two copies of given graph joined at same vertex by t paths of same length for vertex prime labeling. These families are denoted by $G(t\text{-}P_m)$ where t is the number of paths P_m.We refer these graphs when t = 1 as $G(P_m)$. We choose G as W_4 and Gear graph G_4. We also discuss $W_4(t\text{-}P_m)$ G_4which is family of graphs having t-paths of same length between W_4 and G_4.

Key words: *Path, labeling, vertex prime, graph.*

Subject Classification: 05C78

2. INTRODUCTION:

The graphs we consider are finite, connected, and simple and un- directed. We refer F.Harary[4], Dynamic survey of graph labeling [3] for definitions and terminology. Deretsky, Lee, Mitchem Proposed a labeling called as vertex prime labeling of graph.[2].A function f : E(G) → {1,2,..|E|} is such that for any vertex v the gcd of all labels on edges incidentwith v is 1. This is true to all vertices with degree at least 2.The graph that admits vertex prime labeling is called as vertex prime graph. They have shown that all forests, connected graphs, $5C_{2m}$, graph with exactly two components one of which is not odd cycle etc are vertex prime. One should refer A Dynamic survey of graph labeling by Joe Gallian[3]to find further work done in this type of labeling.

3.Preliminaries:

3.1 A wheel graph W_n is obtained by taking a cycle C_n and a new vertex w outside of C_n.W is joined to each vertex of C_n by an edge each. It has 2n edges and n+1 vertices. .In this paper we consider w_4 and related graphs for vertex prime labeling.

3.2 Gear graph Gn is obtained from W_n by inserting a new vertex between consecutive vertex of cycle C_n. It has 2n + 1 vertices and 3n edges.In this paper we consider G_4 and related graphs for vertex prime labeling.

3.3Fusion of vertex. Let G be a (p,q) graph. let$u\neq v$ be two vertices of G. We replace them with single vertex w and all edges incident with u and that with v are made incident with w. If a loop is formed is deleted. The new graph has p-1vertices and at least q-1 edges.[6]

3.4 Let G_1 be (p_1,q_1) and G_2be (p_2,q_2) be any two graphs. We join two graphs $G_1$and G_2 by a path P_m.The resultant graph we denote as $G_1(P_m)G_2$.It has P_1+P_2+m-2 vertices and q_1+q_2+m-1 edges .In this paper we discuss $W_4(P_m)G_4$ for vertex prime labeling.

3.5 The collection of positive numbers will be co prime if

i) It contains the number '1'
ii) It contains two consecutive integers
iii) It contains a prime number and no multiple of it

4. Theorems Proved:

4.1 The graph obtained by joining two copies of W_4 by path Pm i.e. $G=W_4(1-P_m)$ is vertex prime. The point of fusion on both copies is hub vertex of W_4.Further if we change the point where the path is fused with W_4 the resultant graph is also vertex prime.

Proof: We define this graph as : $V_1=\{w_1,w_2,u^i_j/j = 1, 2, 3, 4.$ And $i =1, 2)\}$. These are vertices on two copies of W_4. $V_2= \{v_1, v_2, ..v_m\}$.These are vertices on path P_m.The edge set on pokes is given by $E_1=\{(w_iu^i_j) /j = 1, 2, 3, 4.$ And $i =1, 2)\}$. $E_2 =\{(u^i_ju^i_{j+1})/j = 1, 2, 3, 4.$where j+1 is taken modulo 4,and $i = 1,2\}$.E_2 are C_4 -cycle edges on two copies of W_4. $E_3 = \{(v_iv_{i+1})/i = 1, 2, .., m-1.\}$ Since we take the paths at hub w_i we have $w_1 = v_1$ and $w_2 = v_m$. Thus $V(G)=V_1UV_2$. $E(G) =E_1UE_2UE_3$.

Define $f:E(G)\rightarrow\{1, 2, ..,m+15\}$ as follows:

$f(w_iu^i_j)=j$ for $i =1; j=1, 2, 3, 4$

$f(u^i_ju^i_{j+1})=4+j, j=1, 2, 3, 4$ and $i =1$;

$f(v_iv_{i+1})=8+i, i= 1, 2, ..,m-1$;

$f(u^i_ju^i_{j+1})=m+7+j, j=1, 2, 3, 4$ and $i =2$;

$f(w_iu^i_j)=m+11+j$ for $i=2;j=1,2,3,4$.

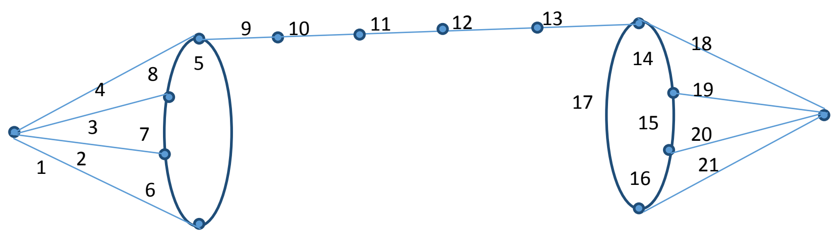

Fig 4.1: A labeled copy of vertex prime graph $W_4(P_6)$Edge labels are shown.P6 is attached at cycle vertex of W_4)

If we shift the point of fusion of P_m with W_4 say to a vertex on outer cycle of W_4, u^1_jandu^2_j for some j =1, 2, 3,4 then the only change will be v_1 will be same as u^1_j and v_m as u^2_j. The function described above will work as vertex prime label.

Thus the graph is vertex prime. #

4.2Theorem: The graph G obtained by joining two copies of W_4 by paths Pm between every pair of like vertex on two copies of W_4(given by G_1 and G_2) is vertex prime. These are paths (ug(u)) where g(u) is image of u under any automorphism g on G_4.Here u and g(u) are like vertices.

Proof:

We start with two vertex prime labeled copies of W_4. Furtherthere will be five different paths from G_1 to G_2.We define this graph as : $V_1=\{w_1, w_2, u^i_j/j = 1, 2, 3, 4.$ And $i =1, 2)\}$. These are vertices on two copies of W_4. $V_2= \{v_{i,1}, vi,_2, .. v_{i,m}\}$.These are path ($P_m$) vertices from vertex $v_{i,1}$ on G_1 to $v_{i,m}$ on G_2. When the path will be (w_1w_2) then w_1 will be $v_{i,1}$and w_2 will be $v_{i,m.}$ The edge set on pokes is given by $E_1=\{(w_iu^i_j) /j = 1, 2, 3, 4.;i =1, 2)\}$. $E_2=\{(u^i_ju^i_{j+1})/j = 1, 2, 3, 4.$where j+1 is taken modulo 4}. E_2 are C_4 -cycle edges on two copies of W_4. $E_3 = \{(v_iv_{i+1})/i = 1, 2, .., m-1.\}$. If we take the paths between hub w_i we have $w_1 = v_1$ and $w_2 = v_m$.Thus $V(G)=V_1UV_2$. $E(G) =E_1UE_2UE_3$.
Define $f:E(G)\rightarrow\{1, 2, ..q\}$ as follows where $q= |E(G)|$:

$f(w_iu^i_j)=j$ for $i =1; j=1, 2, 3, 4$

$f(u^i_ju^i_{j+1})=4+j, j=1, 2, 3, 4$ and $i =1$;

$f(u^i_ju^i_{j+1})=8+j, j=1, 2, 3, 4$ and $i =2$;

$f(w_iu^i_j)=12+j$ for $i=2;j=1,2,3,4$.

At this stage we have completed the vertexprime labeling of both copies of W_4.$f(v_{i,j}v_{i,j+1})=16+i(m-1)+j$, i= 1, 2, 3, 4, 5 ; j = 1, 2, .. (m-1)

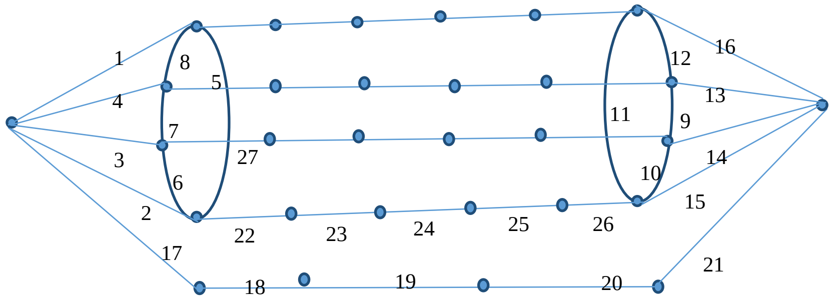

Fig 4.2: A labeled copy of vertex prime graph with 5 pathspathson W_4. Edge labels are shown.

Thus the graph is vertex prime. #

4.3Theorem:

The graph obtained by joining two copies of Gear graph G_4(say G' and G") by path P_m i.e. $G=G_4(P_m)$ is vertex prime. The point of fusion on both copies is hub vertex of G_4. Further if we change the point where the path is fused with G_4 the resultant graph is also vertex prime.

Proof:

We define this graph as : $V_1=\{w_1, w_2, u^i_j/j = 1, 2, ...8$ And $i =1, 2)\}$. These are vertices on two copies of G_4. $V_2= \{v_1, v_2, ..v_m\}$.These are vertices on path P_m. The edge set on pokes is given by $E_1=\{(w_iu^i_j)$ /j = 1, 3, 5, 7. And i =1, 2)}. $E_2=\{(u^i_ju^i_{j+1})$/j = 1, 2, .., 8.where j+1 is taken modulo 8,i=1,2.}. E_2 are C_8 -cycle edges on two copies of G_4. $E_3 = \{(v_iv_{i+1})$/i = 1, 2, .., m-1.} Since we take the paths fused at hub w_i we have $w_1 = v_1$ and $w_2 = v_m$.Thus $V(G)=V_1UV_2$. E(G) $E_1UE_2UE_3$.

Define f:E(G)→{1, 2, ..,m+23} as follows:

$f(w_iu^i_j)$=t for i=2;j=1,3,5,7 and t is such that j = 2t-1

These are labels on pokes of G'

$f(u^i_ju^i_{j+1})$=4+j, j=1, 2, .., 8 and i =1; These are labels on cycle C_8 of G'
$f(v_iv_{i+1})$=12+i, i= 1, 2, ..,m-1;These are labels on path(v_iv_m)
$f(u^i_ju^i_{j+1})$=m+11+j, j=1, 2, .., 8 and i =2;These are labels on cycle C_8 of G"
$f(w_iu^i_j)$=m+19+t for i=2;j=1,3,5,7 and t is such that j = 2t-1

These are labels on pokes of G".

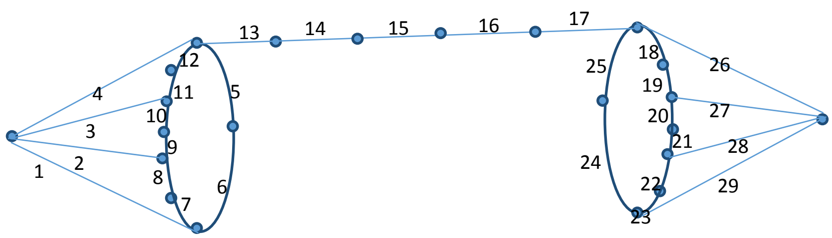

Fig 4.3: A labeled copy of vertex prime graph $G_4(1-P_6)$Edge labels are shown.

If we change the end vertices of path from $v_1 = w_1$ to any vertex on cycle C_8 of G' and $v_m = w_2$ to any vertex on cycle C_8 of G" The above labeling function f will work as it is and the resultant graph will be vertex prime.

Thus the graph is vertex prime. #

4.4 Theorem:

The graph G obtained by joining two copies of G_4(given by G_1 and G_2) by paths P_m between every pair of like vertices G_4 is vertex prime. These are paths (ug(u)) where g(u) is image of u under auto morphism g on G_4.

Proof:

We start with two vertex prime labeled copies of G_4. Furtherthere will be 8 different paths from G_1 to G_2.

We define this graph as : $V_1 = \{w_1, w_2, u^i_j / j = 1, 2, ., 8.$ and $i = 1, 2)\}$. These are vertices on two copies of G_4. $V_2 = \{v_{i,1}, vi,_2, .. v_{i,m}\}$.These are path ($P_m$) vertices from vertex $v_{i,1}$ on G_1 to $v_{i,m}$ on G_2. When the paths will be (w_1w_2) then w_1 will be $v_{i,1}$and w_2 will bev$_{i,m.}$ The edge set on pokes is given by $E_1 = \{(w_iu^i_j)$ /j = 1, 3, 5, 7. And i =1, 2)\}. $E_2=\{(u^i_ju^i_{j+1})/j = 1, 2, .., 8$ where j+1 is taken modulo 4\}. E_2 are C_8 -cycle edges on each copy of G_4. $E_3 = \{(v_iv_{i+1})/i = 1, 2, .., m-1.\}$. Thus G has 9(m-1) edges. FurtherV(G)=V_1UV_2 and E(G) =E_1UE_2UE_3.

Define f:E(G)→{1, 2, ..,q)} as follows where q =| E(G)|
$f(w_iu^i_j)$=t for i=2;j=1,3,5,7 and t is such that j = 2t-1
$f(u^i_ju^i_{j+1})$=4+j, j=1, 2, .., 8 and i =1;
$f(u^i_ju^i_{j+1})$=12+j, j=1, 2, .., 8 and i =2;
$f(w_iu^i_j)$=20+j+t for i=2;j=1,3,5,7 and t is such that j = 2t-1

At this stage we have completed the vertex prime labeling of both copies of G4. $f(v_{i,j}v_{i,j+1})$=24+i(m-1)+j, i= 1, 2, ..,12 ; j = 1, 2, .. (m-1), for any path (ug(u)) where u = $v_{i,1}$ and g(u) = $v_{i,m}$.

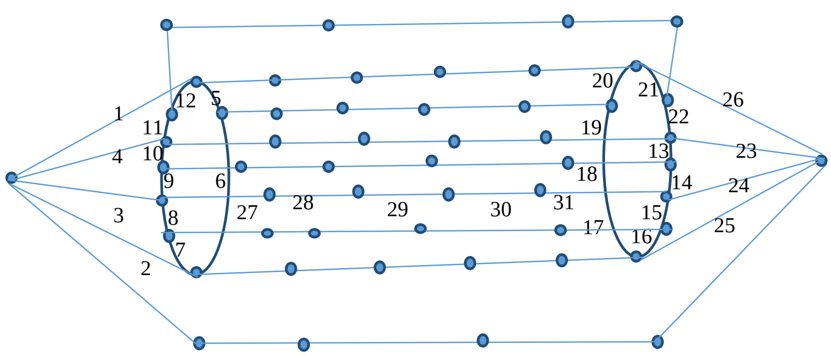

Fig 4.4: A labeled copy of vertex prime graph with 9 pathsP_6 on G_4.A few of the edge labels are shown.

Thus the graph is vertex prime. #

4.5Theorem: Let G be a graph obtained from W_4 and G_4 by joining their hub by a path P_m is vertex prime. If we shift the end points of path Pm from hub to any other vertex on respective graph then also the resultant graph is vertex prime.

Proof:

We define the graph G as follows: $V_1=\{w_1, w_2, u^i_j/j = 1, 2, 3, 4.$ for i =1 and are vertices on W_4 and i = 2 then vertices are on G_4.Further if i=2 then j =1,2, .,,8)}. $V_2 = \{v_1, v_2, ..v_m\}$.These are vertices on path P_m. The edge set on pokes is given by $E_1=\{(w_iu^i_j)$ /j = 1, 2, 3, 4; when i =1for pokes of W_4 and when i =2 these are pokes of G_4 and in that case j = 1,3,5,7)}. $E_2 =\{(u^1_ju^1_{j+1})$ /j = 1, 2, 3, 4.where j+1 is taken modulo 4}. These are cycle vertices of W_4.$E_3 = \{(v_iv_{i+1})/i = 1, 2,$,, m-1.} are edges on path P_m. $E_2' = \{(u^2_ju^2_{j+1})$ /j = 1, 2, ,..8; where j+1 is taken modulo 8}. Care should be taken thatedge ($u^2_1u^2_2$)- the C_8 edge should be adjacent to last edge ($v_{m-1}v_m$) of path incident to C_8. E_2' are C_8 -cycle edges on G_4. Since we take the paths at hub w_i we have $w_1 = v_1$ and $w_2 = v_m$ which is the hub of G4.Thus $V(G)=V_1UV_2$. And $E(G) =E_1UE_2UE2'UE_3$.

Define f:E(G)→{1, 2, ..,m+19} as follows:

$f(w_iu^1_j)=j$ for i =1; j=1, 2, 3, 4

$f(u^1_ju^1_{j+1})=4+j$, j=1, 2, 3, 4
$f(v_jv_{j+1})=8+j$; j = 1,2, ,m-1.
$f(u^2_ju^2_{j+1})=m+7+j$, j=1, 2, .., 8

$f(w_iu^i_j)=m+15+t$ for i=2;j=1, 3,5,7 and t is such that j = 2t-1

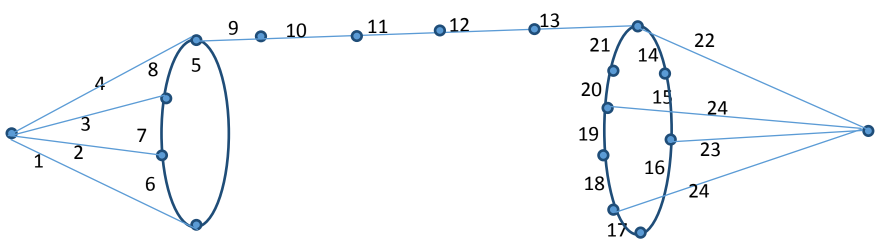

Fig 4.5: A labeled copy of vertex prime graph $W_4(P_6)G_4$Edge labels are shown.

Ifwe change the end vertices of path P_m then the same function f as above will work as vertex prime label function. Thus the above labeling is independent of end points of P_m.

Thus the graph is vertex prime. #

4.6Theorem:

Hub of W_4 is joined to every vertex of other copy of W_4 by paths P_m. The resultant graph is vertex prime.The result holds even if we change the starting point of path namely hub, to any other vertex on W_4.

Proof:

We define this graph as : $V_1=\{w_1, w_2, u^i_j/j = 1, 2, 3, 4.$ And i =1, 2)}. These are vertices on two copies of W_4. $V_2= \{v_{i,1}, v_{i,2},$ .. $v_{i,m}$ where i = 1,2..,5 for five paths ends at five different vertices of W_4 }.These are vertices on path Pm. The edge set on pokes is given by $E_1 = \{(w_iu^i_j)$ /j = 1, 2, 3, 4. And i =1, 2)}. $E_2 = \{(u^1_ju^1_{j+1})$/j = 1, 2, 3, 4.where j+1 is taken modulo 4 }U$\{(u^2_ju^2_{j+1})$/j = 1, 2, 3, 4.where j+1 is taken modulo 4 }.

E_2 are C_4 -cycle edges on two copies of W_4. $E_3 = \{(v_{i,j}v_{i,j+1})$/j = 1, 2, .., m-1; i = 1, 2,..5 }. Since we take the path starting at hub w_i we have $w_1 = v_{i,1}$.Thus $V(G)=V_1UV_2$. $E(G) =E_1UE_2UE_3$.

Define f:E(G)→{1, 2, ..,q} as follows q= |E(G)|:
$f(w_iu^i_j)=j$ for i =1; j=1, 2, 3, 4
$f(u^i_ju^i_{j+1})=4+j$, j=1, 2, 3, 4 and i =1;

$f((v_{i,j}v_{i,j+1}))=8+(m-1)(i-1)+j$, j= 1, 2, ..,m-1;i=1, 2..5.

$f(u^i_j u^i_{j+1})=5m+3+j$, j=1, 2, 3, 4 and i =2;

$f(w_i u^i_j)=5m+7+j$ for i=2;j=1,2,3,4.

If we change starting point of all paths from hub to any other vertex on W_4 still above labeling function f will serve as vertex prime label.

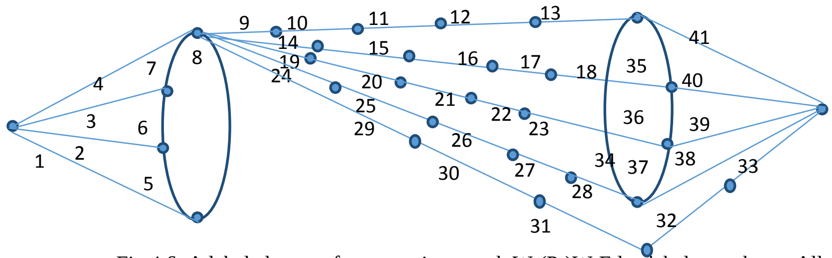

Fig 4.6: A labeled copy of vertex prime graph $W_4(P_6)W_4$Edge labels are shown.All paths starts from a cycle vertex of W_4

Thus the graph is vertex prime. #

Conclusions:

We have defined a new family of graphs $G_1(t\text{-}p_m)G_2$.When t =1 write $G_1(p_m)G_2$.When G1 = W4and G2 = G4 we have shown that graph is vertex prime. Further The graphs are vertex prime independent of starting and end point of path P_m.We have also shown that if we take 5 paths starting at hub vertex of W_4 and each one ending at different vertex of other copy of W_4, the resultant graph is vertex prime. If we change the starting vertex from hub to any other vertex of W_4 still we have shown that the same function works as vertex prime label. It is necessary to investigate the similar graphs for vertex prime labeling.

References:

[1] Mukund V. Bapat Ph.D. thesis, university of Mumbai,India,2004.

[2] T. Deretsky, S. M. Lee, and J. Mitchem, On vertex prime labelings of graphs, in Graph Theory, Combinatorics and Applications Vol. 1, J. Alavi, G. Chartrand, O. Oellerman, and A. Schwenk, eds., Proceedings 6th International Conference Theory and Applications of Graphs (Wiley, New York, 1991) 359-369.

[3] Joe Gallian Dynamic survey of graph labeling 2016

[4] Harary,GraphTheory,Narosa publishing ,New Delhi

[5] Yilmaz,Cahit ,E-cordial graphs,Ars combina,46,251-256.

[6] Introduction to Graph Theory by D. WEST, Pearson Education Asia. 1 Mukund V. Bapat, Hindale,Devgad,Sindhudurg,Maharashtra India: 416630