Lie Symmetries of (2+1)-dimensional Modified Equal Width Wave Equation

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Abstract

We establish the symmetry reductions of (2+1)- Dimensional Modified Equal Width Wave Equation is subjected to the Lie's classical method. Classification of its symmetry algebra into one- and two-dimensional subalgebras are carried out in order to facilitate its reduction systematically to (1+1)-dimensional PDEs and then to first or second-order ODEs.
Keywords: Nonlinear PDE,Lie's Classical Method,Lies Algebra and Symmetry Reductions.

Introduction

A simple model equation is the Korteweg-de Vries (KdV) equation [8]

$$v_t + 6vv_x + \delta v_{xxx} = 0 \ , \tag{1}$$

which describe the long waves in shallow water. Its modified version is,

$$u_t - 6u^2 u_x + u_{xxx} = 0 \tag{2}$$

and again there is Miura tramsformation [7]

$$v = u^2 + u_x \ , \tag{3}$$

between the KdV equation (1) and its modified version (2).

In 2002, Liu and Yang [6] studied the bifurcation properties of generalized KdV equation (GKdVE)

$$u_t + au^n u_x + u_{xxx} = 0 \ , \quad a \ \in R \ , \ n \ \in \ Z^+ \ . \tag{4}$$

Gungor and Winternitz [10] transformed the Generalized Kadomtsev-Petviashvili Equation (GKPE)

$$(u_t + p(t)uu_x + q(t)u_{xxx})_x + \sigma(y,t)u_{yy} + a(y,t)u_y + b(y,t)u_{xy}$$

$$+c(y,t)u_{xx} + e(y,t)u_x + f(y,t)u + h(y,t) = 0 \ , \tag{5}$$

to its canonical form and established conditions on the coefficient functions under which (5) has an infinite dimensional symmetry group having a Kac-Moody-Virasoro structure.

In [13], they carried out the symmetry analysis of Variable Coefficient Kadomtsev Petviashvili Equation (VCKP) in the form,

$$(u_t + f(x,y,t)uu_x + g(x,y,t)u_{xxx})_x + h(x,y,t)u_y = 0 \ .$$

Burgers' equation $u_t + uu_x = \gamma u_{xx}$, is the simplest second order NLPDE which balances the effect of nonlinear convection and the linear diffusion. In this chapter, we discuss the symmetry reductions of the (2+1)-dimensional modified Equal Width Wave equation as,

$$u_t + u^3 u_x - u_{xxt} - u_{yyt}. \tag{6}$$

Our intention is to show that equation (6) admits a four-dimensional symmetry group and determine the corresponding Lie algebra, classify the one- and two-dimensional subalgebras of the symmetry algebra of (6), in order to reduce (6) to (1+1)-dimensional PDEs and then to ODEs. We shall establish that the symmetry generators form a closed Lie algebra and this allowed us to use the recent method due to Ahmad, Bokhari, Kara and Zaman [11] to successively reduce (6) to (1+1)-dimensional PDEs and ODEs with the help of two-dimensional Abelian and non-Abelian solvable subalgebras.This chapter is organised as follows: First, we determine the symmetry group of (6) and write down the associated Lie algebra. secondly, we consider all one-dimensional subalgebras and obtain the corresponding reductions to (1+1)-dimensional PDEs. Next, we show that the generators form a closed Lie algebra and use this fact to reduce (6) successively to (1+1)- dimensional PDEs and ODEs. Finally, we summarises the conclusions of the present work.

The Symmetry Group and Lie Algebra of modified Equal Width wave equation

If (6) is invariant under a one parameter Lie group of point transformations (Bluman and Kumei [1-3], Olver [4])

$$x^* = x + \epsilon\, \xi(x,y,t;u) + O(\epsilon^2)\ , \tag{7}$$
$$y^* = y + \epsilon\, \eta(x,y,t;u) + O(\epsilon^2)\ , \tag{8}$$
$$t^* = t + \epsilon\, \tau(x,y,t;u) + O(\epsilon^2)\ , \tag{9}$$
$$u^* = u + \epsilon\, \phi(x,y,t;u) + O(\epsilon^2)\ . \tag{10}$$

Then the third Prolongation $Pr^3(V)$ of the corresponding vector field

$$V = \xi(x,y,t;u)\frac{\partial}{\partial x} + \eta(x,y,t;u)\frac{\partial}{\partial y} + \tau(x,y,t;u)\frac{\partial}{\partial t} + \phi(x,y,t;u)\frac{\partial}{\partial u}, \tag{11}$$

satisfies

$$pr^3(V)\Omega(x,y,t;u)|_{\Omega(x,y,t;u=0} = 0. \tag{12}$$

The determining equations are obtained from (12) and solved for the infinitesimals ξ, η, τ and ϕ. They are as follows

$$\xi = k_1\ , \tag{13}$$

$$\begin{aligned} \eta &= k_4 \ , && (14)\\ \tau &= k_2 - 3k_3 t \ , && (15)\\ \phi &= k_4 u \ . && (16) \end{aligned}$$

At this stage, we construct the symmetry generators corresponding to each of the constants involved.
Totally there are three generators given by

$$\begin{aligned} V_1 &= \partial_x \ ,\\ V_2 &= \partial_t,\\ V_3 &= -3t\partial_t + u\partial_u\\ V_2 &= \partial_y && (17) \end{aligned}$$

The symmetry generators found in Eq.(17) form a closed Lie Algebra whose commutation table is shown below.
Table 1 Commutator Table

$[V_i, V_j]$	V_1	V_2	V_3	V_4
V_1	0	0	0	0
V_2	0	0	$-3V_2$	0
V_3	0	0	0	0
V_4	0	0	0	0

The commutation relations of the Lie algebra, determined by V_1, V_2, V_3 and V_4 are shown in the above table.

For this four-dimensional Lie algebra the commutator table for V_i is a $(4 \otimes 4)$ table whose $(i, j)^{th}$ entry expresses the Lie Bracket $[V_i, V_j]$ given by the above Lie algebra L. The table is skew-symmetric and the diagonal elements all vanish. The coefficient $C_{i,j,k}$ is the coefficient of V_i of the $(i, j)^{th}$ entry of the commutator table. The Lie algebra L is solvable.

In the next section, we derive the reduction of (6) to PDEs with two independent variables and ODEs. These are four one-dimensional Lie subalgebras

$$L_{s,1} = \{V_1\} \ , \ L_{s,2} = \{V_2\} \ , \ L_{s,3} = \{V_3\} \ , \ \ L_{s,4} = \{V_4\}$$

and corresponding to each one-dimensional subalgebras we may reduce (6) to a PDE with two independent variables.

Further reductions to ODEs are associated with two-dimensional subalgebras.
It is evident from the commutator table that there is one two-dimensional solvable non-abelian subalgebras. And there are five two-dimensional Abelian subalgebras, namely,

$$L_{A,1} = \{V_1, V_2\}, \ L_{A,2} = \{V_1, V_3\} \ , \ L_{A,3} = \{V_1, V_4\} \ L_{A,4} = \{V_2, V_4\} \ L_{A,5} = \{V_3, V_4\} \ . \quad (18)$$

Reductions of (2+1)-dimensional Modified Equal Width wave equation by One-Dimensional Subalgebras

Case 1 : $V_1 = \partial_x$.
The characteristic equation associated with this generator is

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{0} \ . \tag{19}$$

We integrate the characteristic equation to get three similarity variables,

$$y = r, \quad t = s \quad and \quad u = W(r,s) \ . \tag{20}$$

Using these similarity variables in Eq.(6) can be recast in the form

$$W_s - W_{rrs} = 0. \tag{21}$$

Case 2 : $V_2 = \partial_t$.
The characteristic equation associated with this generator is

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{1} = \frac{du}{0} \ . \tag{22}$$

Following the standard procedure we integrate the characteristic equation to get three similarity variables,

$$x = r, \quad y = s \quad and \quad u = W(r,s) \ . \tag{23}$$

Using these similarity variables in Eq.(6) can be recast in the form

$$W^3 W_r = 0. \tag{24}$$

Case 3 : $V_3 = -3t\partial_t \ + upartial_u$.
The characteristic equation associated with this generator is

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{-3t} = \frac{du}{u} \ . \tag{25}$$

Following standard procedure we integrate the characteristic equation to get three similarity variables,

$$x = r, \quad y = s \quad and \quad u = W^{1/3}(r,s)t^{-1/3} \ . \tag{26}$$

Using these similarity variables in Eq.(6) can be recast in the form

$$W^{1/3} - W^{1/3}W_r + (2/9)W^{-5/3}({W_r}^2 + {W_s}^2) - (1/3)W^{-2/3}(W_{rr} + W_{ss}) = 0. \tag{27}$$

Case 4 : $V_4 = \partial_y$.
The characteristic equation associated with this generator is

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dt}{0} = \frac{du}{0} \ . \tag{28}$$

Following standard procedure we integrate the characteristic equation to get three similarity variables,

$$x = r, \quad t = s \quad and \quad u = W(r, s) \ . \tag{29}$$

Using these similarity variables in Eq.(6) can be recast in the form

$$W_s + W^3(W_r) - W_{rrs} = 0. \tag{30}$$

Reductions of (2+1)-dimensional modified Equal Width wave equationby Two-Dimensional Abelian Subalgebras

Case I : Reduction under V_1 and V_2 .
From Table 1 we find that the given generators commute $[V_1, V_2] = 0$. Thus either of V_1 or V_2 can be used to start the reduction with. For our purpose we begin reduction with V_1. Therefore we get Eq.(20) and Eq.(21).
At this stage, we express V_2 in terms of the similarity variables defined in (20). The transformed V_2 is

$$\tilde{V_2} = \partial s. \tag{31}$$

The characteristic equation for $\tilde{V_2}$ is

$$\frac{dr}{0} = \frac{ds}{1} = \frac{dW}{0} \ .$$

Integrating this equation as before leads to new variables

$$r = \zeta \ \text{ and } \ W = R(\zeta),$$

which reduce Eq.(21) to

$$-R_(\zeta\zeta) + R(\zeta) = 0 \ . \tag{32}$$

Case II: Reduction under V_1 and V_3 .
From Table 1 we find that the given generators commute $[V_1, V_3] = 0$. Thus either of V_1 or V_3 can be used to start the reduction with. For our convenience we begin reduction with V_3. At this stage, we express V_1 in terms of the similarity variables defined in Eq.(26). The transformed V_1 is

$$\tilde{V_1} = \partial_r. \tag{33}$$

The characteristic equation for $\tilde{V_1}$ is

$$\frac{dr}{1} = \frac{ds}{0} = \frac{dW}{0}.$$

Integrating this equation as before leads to new variables

$$s = \zeta \ \text{ and } \ W = R(\zeta) \ .$$

which reduce Eq.(27) to

$$R(\zeta)^{1/3} + (2/9)R^{-5/3}(R_\zeta)^2 - (1/3)R^{-2/3}(R_{\zeta\zeta}) = 0. \tag{34}$$

Case III : Reduction under V_1 and V_4 .
From Table 1 we find that the given generators commute $[V_1, V_4] = 0$. Thus either of V_1 or V_4 can be used to start the reduction with. For our convenience we begin reduction with V_1. Therefore we get Eq.(20) and Eq.(21).
At this stage, we express V_1 in terms of the similarity variables defined in Eq.(20). The transformed V_4 is

$$\tilde{V_4} = \partial_r \ . \tag{35}$$

The characteristic equation for $\tilde{V_4}$ is $\mathrm{dr}_{\overline{1=\frac{ds}{0}=\frac{dW}{0}}}$. Integrating this equation as before leads to new variables

$$s = \zeta \ \text{ and } \ W = R(\zeta) \ .$$

which reduce Eq.(21) to

$$R_(\zeta) = 0 \ . \tag{36}$$

Case IV : Reduction under V_2 and V_4 .
From Table 1 we find that the given generators commute $[V_2, V_4] = 0$. Thus either of V_2 or V_4 can be used to start the reduction with. For our convenience we begin reduction with V_2. Therefore we get Eq.(23) and Eq.(24).
At this stage, we express V_4 in terms of the similarity variables defined in Eq.(23). The transformed V_4 is

$$\tilde{V_4} = \partial_s \ . \tag{37}$$

The characteristic equation for $\tilde{V_4}$ is $\mathrm{dr}_{\overline{0=\frac{ds}{1}=\frac{dW}{0}}}$. Integrating this equation as before leads to new variables

$$r = \zeta \ \text{ and } \ W = R(\zeta) \ .$$

which reduce Eq.(24) to

$$R^3 R_(\zeta) = 0 \ . \tag{38}$$

Case V : Reduction under V_3 and V_4 .
From Table 1 we find that the given generators commute $[V_3, V_4] = 0$. Thus either of V_3 or V_4 can be used to start the reduction with. For our convenience we begin reduction with V_3.

Therefore we get Eq.(26) and Eq.(27).
At this stage, we express V_4 in terms of the similarity variables defined in Eq.(26). The transformed V_4 is

$$\tilde{V_4} = \partial_s \ . \tag{39}$$

The characteristic equation for $\tilde{V_4}$ is $\mathrm{dr}_{\overline{0=\frac{ds}{1}=\frac{dW}{0}}}$. Integrating this equation as before leads to new variables

$$r = \zeta \ \text{ and } \ W = R(\zeta) \ .$$

which reduce Eq.(27) to

$$R^{1/3} - R^{1/3}R_(\zeta) + (2/9)R^{-5/3}R_(\zeta)^2 - (1/3)R^{-2/3}R_(\zeta\zeta) = 0 \ . \tag{40}$$

Conclusions:
In this chapter, A $(2+1)$-dimensional Modified Equal width wave equation, $u_t + u^2 u_x - (u_{xxt} + u_{yyt}) = 0$ is subjected to Lie's classical method. Equation (6) admits a three-dimensional symmetry group. It is established that the symmetry generators form a closed Lie algebra. Classification of symmetry algebra of (6) into one- and two-dimensional subalgebras is carried out. Systematic reduction to (1+1)-dimensional PDE and then to first order ODEs are performed using one-dimensional and two-dimensional solvable Abelian subalgebras.

References

1. G. Bluman, S. Kumei, *Symmetries and Differential Equations*, Springer-Verlag, New York, (1989).

2. S. Padmasekaran, S.Rajeswari and G. Sivagami,*Similarity Solution of Semilinear Parabolic Equations with Variable Coefficients*,IJMAA,Volume 4, Issue 3A (2016), 201209.

3. S.Padmasekaran and S.Rajeswari,*Solitons and Exponential Solutions for a Nonlinear (2+1)dim PDE*, IJPAM,Volume 115 No. 9 2017, 121-130.

4. S. Padmasekaran and S. Rajeswari, *Lie Symmetries of (2+1)dim PDE*IJMTT,Volume 51 No. 6,11-2017.

5. S. Padmasekaran and S.Rajeswari,*Solitons and Exponential Solutions for a Nonlinear (2+1)dim PDE*, IJPAM,Volume 115 No. 9 2017, 121-130.

6. S. Padmasekaran and S. Rajeswari, *Lie's Symmetries of (2+1)dim PDE*IJMTT,Volume 51 No. 6,11-2017.

7. P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, (1986).

8. A. Ahmad, Ashfaque H. Bokhari, A. H. Kara, F. D. Zaman, *Symmetry Classifications and Reductions of Some Classes of (2+1)-Nonlinear Heat Equation*, J. Math. Anal. Appl., **339**: 175-181,(2008).

9. Z. Liu and C. Yang ,*The application of bifurcation method to a higher-order KdV equation*,J. Math. Anal. Appl., **275**: 1-12, (2002).

10. R. M. Miura, *Korteweg-de Vries equations and generalizations. A remarkable explicit nonlinear transformation*, I.Math. Phys. **9**: 1202-1204, (1968).

11. D. J. Korteweg and G. de Vries, *On the Chans of Form of Long Waves Advancing in a Rectangular canal, and On a New type of Long Stationary Waves*, Philosophical Magazine **39** (1985), 422-443.

12. F. Gungor and P. Winternitz, *Generalized Kadomtsev Petviashvili equation with an infinitesimal dimensional symmetry algebra*, J. Math. Anal. **276** (2002), 314-328.

13. F. Gungor and P. Winternitz, *Equaivalence Classes and Symmetries of the Variable Coefficient Kadomtsev Petviashvili Equation*, Nonlinear Dynamics **35** (2004), 381-396.

14. P. J. Olver, *Applications of Lie Groups to Differential Equations*, Graduate Texts in Mathematics, Springer-Verlag, New York, 1986.

15. N.H. Ibragimov, *Transformation Groups Applied to Mathematical Physics*, Reidel Publishing Company, The Netherlands, 1985.

16. R.E. Grundy, *Similarity solutions of the nonlinear diffusion equation*, Quart. Appl. Math.,37 (1979) 259-280.

17. D.L. Hill, J.M. Hill, *Similarity solutions for nonlinear diffusion- further exact solutions,* J. Eng. Math., 24 (1990) 109-124.

18. J.M. Hill, *Similarity solutions for nonlinear diffusion- a new integration procedure,* J. Eng.Math., 23 (1989) 141-155.