Modelling of Biserial Bulk Queue Network Linked With Common Server

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Abstract— *Queuing theory continues to be one of the most extensive theories of stochastic models. Advanced theoretical models are being developed through innovative analytical study with vast applications. The present paper deals with the modelling of biserial bulk queue network with fixed batch size and connected with a common server. Performance measure of the model has been analysed in stochastic environment. Numerical illustration is provided to understand the model in a better way.*
Keywords — *Batch arrival, queue characteristics, stochastic environment.*

I. INTRODUCTION

The initial study of queuing theory was carried out by A.K. Erlang [1]. The multi input idea in queuing theory was introduced by Pandey [2]. Jackson [3] derived the differential difference equations and obtained the steady state equations. The transient solution of a queue network model consisting of two queues in series was studied by O'Brein [4] with poisson input and exponential holding time. Queuing problems with batch arrival was studied by Suzuki [5]. The basic concept of biserial queuing system into the theory of queues was introduced by Maggu ([6], [7]). Further the transient analysis of serial queues under service parameter constraints was discussed by Maggu [8]. Two biseries queues with batch arrival at each stage were studied by Hafiz Noor Mohammad et.al. [9].
Singh T.P. and Kumar Vinod et.al. [10] studied the transient behaviour of a queuing network with parallel biseries queues. Further Singh T.P. and Gupta Deepak [11] analysed a queue network model comprised of biserial and parallel channel linked with a common server. The queue network model having batch arrival with threshold effect was studied by Mittal Meenu et.al. [12]. Later on priority queue model along intermediate queue under fuzzy environment with application was studied by Mittal Meenu et.al. [13] and Singh T.P., Kusum & Gupta Deepak [14] studied the feedback queue model where service rate were assumed propotional to queue numbers .The present paper is further an extension of the study made by Hafiz Noor Mohammad et.al. [9] in the sense that this paper deals with modelling of biserial bulk queue network with fixed size batch arrival and connected with a common server. Performance measure of the model has been analysed in stochastic environment. To understand the model in a better way numerical illustration is provided.

II. Description of the model

The queue network model in the problem consists of three service channels S_1, S_2, and S_3.The S_1 and S_2 are biseries service channels with queues q_1 and q_2 in front of these channels respectively. The third service channel S_3 is commonly linked with these two service channels.

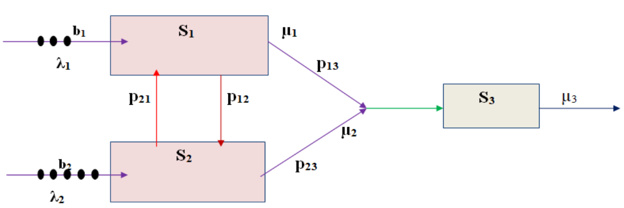

Fig. 1 Biserial bulk queue network model linked with common server

The customers demanding service arrive in batches of fixed sizes b_1 and b_2 with arrival rates λ_1 and λ_2 under poisson assumption and joins the queues q_1 and q_2 respectively. The mean service rates at these service channels have been assumed as μ_1, μ_2 and μ_3 respectively and are exponentially distributed.

Customers coming at rate λ_1 after completion of service at S_1 will either join S_2 or S_3 with the probabilities p_{12} or p_{13} such that $p_{12} + p_{13} = 1$ and those coming at the rate λ_2 after completion of service at S_2 will join either S_1 or S_3 with the probabilities p_{21} or p_{23} such that $p_{21} + p_{23} = 1$ respectively. After completion of service at S_3 the customer is allowed to depart from the system.

A. Practical Situations

Many practical situations of model arise in banking system, administrative setup, club management, handling of children park, supermarket and production management etc. For example, suppose a game club consists of three sections S_1, S_2 & S_3. The sections S_1 and S_2 provide the facility to play the games. Both the games can be played only with a team of fixed size say b_1 & b_2 respectively. The customers entering into the club either may join the section S_1 or S_2. Also there is a chance that a customer can either move from one game to another or directly towards the section S_3 for security measures.

B. Mathematical Analysis

The differential difference equations in transient form are as follows:

$$P'_{n_1.n_2.n_3}(t) = -(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3)P_{n_1.n_2.n_3}(t) + \lambda_1 P_{n_1-b_1.n_2.n_3}(t) + \lambda_2 P_{n_1.n_2-b_2.n_3}(t) + \mu_1 p_{12} P_{n_1+1.n_2-1.n_3}(t) + \mu_1 p_{13} P_{n_1+1.n_2.n_3-1}(t) + \mu_2 p_{21} P_{n_1-1.n_2+1.n_3}(t) + \mu_2 p_{23} P_{n_1.n_2+1.n_3-1}(t) + \mu_3 P_{n_1.n_2.n_3+1}(t) \quad (1)$$

for $(n_1 > b_1,\ n_2 > b_2,\ n_3 > 0)$

$$P'_{n_1.n_2.n_3}(t) = -(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3)P_{n_1.n_2.n_3}(t) + \lambda_1 P_{n_1-b_1.n_2.n_3}(t) + \mu_1 p_{12} P_{n_1+1.n_2-1.n_3}(t) + \mu_1 p_{13} P_{n_1+1.n_2.n_3-1}(t) + \mu_2 p_{21} P_{n_1-1.n_2+1.n_3}(t) + \mu_2 p_{23} P_{n_1.n_2+1.n_3-1}(t) + \mu_3 P_{n_1.n_2.n_3+1}(t) \quad (2)$$

for $(n_1 > b_1,\ n_2 < b_2,\ n_3 > 0)$

$$P'_{n_1.0.n_3}(t) = -(\lambda_1+\lambda_2+\mu_1+\mu_3)P_{n_1.0.n_3}(t) + \lambda_1 P_{n_1-b_1.0.n_3}(t) + \mu_1 p_{13} P_{n_1+1.0.n_3-1}(t) + \mu_2 p_{21} P_{n_1-1.1.n_3}(t) + \mu_2 p_{23} P_{n_1.1.n_3-1}(t) + \mu_3 P_{n_1.0.n_3+1}(t) \quad (3)$$

for $(n_1 > b_1,\ n_2 = 0,\ n_3 > 0)$

$$P'_{n_1.n_2.0}(t) = -(\lambda_1+\lambda_2+\mu_1+\mu_2)P_{n_1.n_2.0}(t) + \lambda_1 P_{n_1-b_1.n_2.0}(t) + \lambda_2 P_{n_1.n_2-b_2.0}(t) + \mu_1 p_{12} P_{n_1+1.n_2-1.0}(t) + \mu_2 p_{21} P_{n_1-1.n_2+1.0}(t) + \mu_3 P_{n_1.n_2.1}(t) \quad (4)$$

for $(n_1 > b_1,\ n_2 > b_2,\ n_3 = 0)$

$$P'_{n_1.n_2.0}(t) = -(\lambda_1+\lambda_2+\mu_1+\mu_2)P_{n_1.n_2.0}(t) + \lambda_1 P_{n_1-b_1.n_2.0}(t) + \mu_1 p_{12} P_{n_1+1.n_2-1.0}(t) + \mu_2 p_{21} P_{n_1-1.n_2+1.0}(t) + \mu_3 P_{n_1.n_2.0}(t) \quad (5)$$

for $(n_1 > b_1,\ n_2 < b_2,\ n_3 = 0)$

$$P'_{n_1.0.0}(t) = -(\lambda_1+\lambda_2+\mu_1)P_{n_1.0.0}(t) + \lambda_1 P_{n_1-b_1.0.0}(t) + \mu_2 p_{21} P_{n_1-1.1.0}(t) + \mu_3 P_{n_1.0.1}(t) \quad (6)$$

for $(n_1 > b_1,\ n_2 = 0,\ n_3 = 0)$

$$P'_{0.n_2.n_3}(t) \;\; -(\lambda_1+\lambda_2+\mu_2+\mu_3)P_{0.n_2.n_3}(t) + \lambda_2 P_{0.n_2-b_2.n_3}(t) + \mu_1 p_{12} P_{1.n_2-1.n_3}(t) + \mu_1 p_{13} P_{1.n_2.n_3-1}(t) + \mu_2 p_{23} P_{0.n_2+1.n_3-1}(t) + \mu_3 P_{0.n_2.n_3+1}(t) \quad (7)$$

for $(n_1 = 0,\ n_2 > b_2,\ n_3 > 0)$

$$P'_{0.n_2.n_3}(t) = -(\lambda_1+\lambda_2+\mu_2+\mu_3)P_{0.n_2.n_3}(t) + \mu_1 p_{12} P_{1.n_2-1.n_3}(t) + \mu_1 p_{13} P_{1.n_2.n_3-1}(t) + \mu_2 p_{23} P_{0.n_2+1.n_3-1}(t) + \mu_3 P_{0.n_2.n_3+1}(t) \quad (8)$$

for $(n_1 = 0,\ n_2 < b_2,\ n_3 > 0)$

$$P'_{0.0.n_3}(t) = -(\lambda_1+\lambda_2+\mu_3)P_{0.0.n_3}(t) + \mu_1 p_{13} P_{1.0.n_3-1}(t) + \mu_2 p_{23} P_{0.1.n_3-1}(t) + \mu_3 P_{0.0.n_3+1}(t) \quad (9)$$

for $(n_1 = 0,\ n_2 = 0,\ n_3 > 0)$

$$P'_{0.n_2.0}(t) = -(\lambda_1+\lambda_2+\mu_2)P_{0.n_2.0}(t) + \lambda_2 P_{0.n_2-b_2.0}(t) + \mu_1 p_{12} P_{1.n_2-1.0}(t) + \mu_3 P_{0.n_2.1}(t) \quad (10)$$

for $(n_1 = 0,\ n_2 > b_2,\ n_3 = 0)$

$$P'_{0.n_2.0}(t) = -(\lambda_1+\lambda_2+\mu_2)P_{0.n_2.0}(t) + \mu_1 p_{12} P_{0.n_2-1.0}(t) + \mu_3 P_{0.n_2.1}(t) \quad (11)$$

for $(n_1 = 0,\ n_2 < b_2,\ n_3 = 0)$

$$P'_{0.0.0}(t) = -(\lambda_1+\lambda_2)P_{0.0.0}(t) + \mu_3 P_{0.0.1}(t) \quad (12)$$

for($n_1 = 0, n_2 = 0, n_3 = 0$)

$$P'_{n_1.n_2.n_3}(t) = -(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3)P_{n_1.n_2.n_3}(t) + \lambda_2 P_{n_1.n_2-b_2.n_3}(t) + \mu_1 p_{12} P_{n_1+1.n_2-1.n_3}(t) + \mu_1 p_{13} P_{n_1+1.n_2.n_3-1}(t) + \mu_2 p_{21} P_{n_1-1.n_2+1.n_3}(t) + \mu_2 p_{23} P_{n_1.n_2+1.n_3-1}(t) + \mu_3 P_{n_1.n_2.n_3+1}(t) \tag{13}$$

for($n_1 < b_1, n_2 > b_2, n_3 > 0$)

$$P'_{n_1.n_2.n_3}(t) = -(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3)P_{n_1.n_2.n_3}(t) + \mu_1 p_{12} P_{n_1+1.n_2-1.n_3}(t) + \mu_1 p_{13} P_{n_1+1.n_2.n_3-1}(t) + \mu_2 p_{21} P_{n_1-1.n_2+1.n_3}(t) + \mu_2 p_{23} P_{n_1.n_2+1.n_3-1}(t) + \mu_3 P_{n_1.n_2.n_3+1}(t) \tag{14}$$

for($n_1 < b_1, n_2 < b_2, n_3 > 0$)

$$P'_{n_1.0.n_3}(t) = -(\lambda_1+\lambda_2+\mu_1+\mu_3)P_{n_1.0.n_3}(t) + \mu_1 p_{13} P_{n_1+1.0.n_3-1}(t) + \mu_2 p_{21} P_{n_1-1.1.n_3}(t) + \mu_2 p_{23} P_{n_1.1.n_3-1}(t) + \mu_3 P_{n_1.0.n_3+1}(t) \tag{15}$$

for($n_1 < b_1, n_2 = 0, n_3 > 0$)

$$P'_{n_1.n_2.0}(t) = -(\lambda_1+\lambda_2+\mu_1+\mu_2)P_{n_1.n_2.0}(t) + \lambda_2 P_{n_1.n_2-b_2.0}(t) + \mu_1 p_{12} P_{n_1+1.n_2-1.0}(t) + \mu_2 p_{21} P_{n_1-1.n_2+1.0}(t) + \mu_3 P_{n_1.n_2.1}(t) \tag{16}$$

for($n_1 < b_1, n_2 > b_2, n_3 = 0$)

$$P'_{n_1.n_2.0}(t) = -(\lambda_1+\lambda_2+\mu_1+\mu_2)P_{n_1.n_2.0}(t) + \lambda_2 P_{n_1.n_2-b_2.0}(t) + \mu_1 p_{12} P_{n_1+1.n_2-1.0}(t) + \mu_2 p_{21} P_{n_1-1.n_2+1.0}(t) + \mu_3 P_{n_1.n_2.1}(t) \tag{17}$$

for($n_1 < b_1, n_2 < b_2, n_3 = 0$)

$$P'_{n_1.0.0}(t) = -(\lambda_1+\lambda_2+\mu_1)P_{n_1.0.0}(t) + \mu_2 P_{21} P_{n_1-1.1.0}(t) + \mu_3 P_{n_1.n_2.1}(t) \tag{18}$$

for($n_1 < b_1, n_2 = 0, n_3 = 0$)

Taking limit as $t \to \infty$, corresponding steady state equations are as follows:

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3)P_{n_1.n_2.n_3} = \lambda_1 P_{n_1-b_1.n_2.n_3} + \lambda_2 P_{n_1.n_2-b_2.n_3} + \mu_1 p_{12} P_{n_1+1.n_2-1.n_3} + \mu_1 p_{13} P_{n_1+1.n_2.n_3-1} + \mu_2 p_{21} P_{n_1-1.n_2+1.n_3} + \mu_2 p_{23} P_{n_1.n_2+1.n_3-1} + \mu_3 P_{n_1.n_2.n_3+1} \tag{19}$$

for($n_1 > b_1, n_2 > b_2, n_3 > 0$)

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3)P_{n_1.n_2.n_3} = \lambda_1 P_{n_1-b_1.n_2.n_3} + \mu_1 p_{12} P_{n_1+1.n_2-1.n_3} + \mu_1 p_{13} P_{n_1+1.n_2.n_3-1} + \mu_2 p_{21} P_{n_1-1.n_2+1.n_3} + \mu_2 p_{23} P_{n_1.n_2+1.n_3-1} + \mu_3 P_{n_1.n_2.n_3+1} \tag{20}$$

for($n_1 > b_1, n_2 < b_2, n_3 > 0$)

$$(\lambda_1+\lambda_2+\mu_1+\mu_3)P_{n_1.0.n_3} = \lambda_1 P_{n_1-b_1.0.n_3} + \mu_1 p_{13} P_{n_1+1.0.n_3-1} + \mu_2 p_{21} P_{n_1-1.1.n_3} + \mu_2 p_{23} P_{n_1.1.n_3-1} + \mu_3 P_{n_1.0.n_3+1} \tag{21}$$

for($n_1 > b_1, n_2 = 0, n_3 > 0$)

$$(\lambda_1+\lambda_2+\mu_1+\mu_2)P_{n_1.n_2.0} = \lambda_1 P_{n_1-b_1.n_2.0} + \lambda_2 P_{n_1.n_2-b_2.0} + \mu_1 p_{12} P_{n_1+1.n_2-1.0} + \mu_2 p_{21} P_{n_1-1.n_2+1.0} + \mu_3 P_{n_1.n_2.1} \tag{22}$$

for($n_1 > b_1, n_2 > b_2, n_3 = 0$)

$$(\lambda_1+\lambda_2+\mu_1+\mu_2)P_{n_1.n_2.0} = \lambda_1 P_{n_1-b_1.n_2.0} + \mu_1 p_{12} P_{n_1+1.n_2-1.0} + \mu_2 p_{21} P_{n_1-1.n_2+1.0} + \mu_3 P_{n_1.n_2.0} \tag{23}$$

for($n_1 > b_1, n_2 < b_2, n_3 = 0$)

$$(\lambda_1+\lambda_2+\mu_1)P_{n_1.0.0} = \lambda_1 P_{n_1-b_1.0.0} + \mu_2 p_{21} P_{n_1-1.1.0} + \mu_3 P_{n_1.0.1} \tag{24}$$

for($n_1 > b_1, n_2 = 0, n_3 = 0$)

$$(\lambda_1+\lambda_2+\mu_2+\mu_3)P_{0.n_2.n_3} = \lambda_2 P_{0.n_2-b_2.n_3} + \mu_1 p_{12} P_{1.n_2-1.n_3} + \mu_1 p_{13} P_{1.n_2.n_3-1} + \mu_2 p_{23} P_{0.n_2+1.n_3-1} + \mu_3 P_{0.n_2.n_3+1} \tag{25}$$

for($n_1 = 0, n_2 > b_2, n_3 > 0$)

$$(\lambda_1+\lambda_2+\mu_2+\mu_3)P_{0.n_2.n_3} = \mu_1 p_{12} P_{1.n_2-1.n_3} + \mu_1 p_{13} P_{1.n_2.n_3-1} + \mu_2 p_{23} P_{0.n_2+1.n_3-1} + \mu_3 P_{0.n_2.n_3+1} \tag{26}$$

for($n_1 = 0, n_2 < b_2, n_3 > 0$)

$$(\lambda_1+\lambda_2+\mu_3)P_{0.0.n_3} = \mu_1 p_{13} P_{1.0.n_3-1} + \mu_2 p_{23} P_{0.1.n_3-1} + \mu_3 P_{0.0.n_3+1} \tag{27}$$

for($n_1 = 0, n_2 = 0, n_3 > 0$)

$$(\lambda_1+\lambda_2+\mu_2)P_{0.n_2.0} = \lambda_2 P_{0.n_2-b_2.0} + \mu_1 p_{12} P_{1.n_2-1.0} + \mu_3 P_{0.n_2.1} \tag{28}$$

for($n_1 = 0, n_2 > b_2, n_3 = 0$)

$$(\lambda_1+\lambda_2+\mu_2)P_{0.n_2.0} = \mu_1 p_{12} P_{0.n_2-1.0} + \mu_3 P_{0.n_2.1} \tag{29}$$

for($n_1 = 0, n_2 < b_2, n_3 = 0$)

$$(\lambda_1+\lambda_2)P_{0.0.0} = \mu_3 P_{0.0.1} \tag{30}$$

for($n_1 = 0, n_2 = 0, n_3 = 0$)

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3)P_{n_1.n_2.n_3} = \lambda_2 P_{n_1.n_2-b_2.n_3} + \mu_1 p_{12} P_{n_1+1.n_2-1.n_3} + \mu_1 p_{13} P_{n_1+1.n_2.n_3-1}$$

$$+ \mu_2 p_{21} P_{n_1 -1. n_2 +1. n_3} + \mu_2 p_{23} P_{n_1 . n_2 +1. n_3 - 1} + \mu_3 P_{n_1 . n_2 . n_3 +1} \quad (31)$$
for$(n_1 < b_1 , n_2 > b_2, n_3 > 0)$

$$(\lambda_1+\lambda_2+\mu_1 +\mu_2+\mu_3) P_{n_1 \cdot n_2 \cdot n_3} = \mu_1 p_{12} P_{n_1 +1. n_2 -1. n_3} + \mu_1 p_{13} P_{n_1 +1. n_2 . n_3 - 1} +\mu_2 p_{21} P_{n_1 -1. n_2 +1. n_3} + \mu_2 p_{23} P_{n_1 . n_2 +1. n_3 - 1} + \mu_3 P_{n_1 . n_2 . n_3 +1} \quad (32)$$
for$(n_1 < b_1 , n_2 < b_2, n_3 > 0)$

$$(\lambda_1+\lambda_2+\mu_1 +\mu_3) P_{n_1 \cdot 0 \cdot n_3} = \mu_1 p_{13} P_{n_1 +1. 0. n_3 - 1} + \mu_2 p_{21} P_{n_1 -1. 1. n_3} + \mu_2 p_{23} P_{n_1 . 1. n_3 - 1} + \mu_3 P_{n_1 . 0. n_3 +1} \quad (33)$$
for$(n_1 < b_1 , n_2 = 0, n_3 > 0)$

$$(\lambda_1+\lambda_2+\mu_1 +\mu_2) P_{n_1 \cdot n_2 \cdot 0} = \lambda_2 P_{n_1 . n_2 - b_2 . 0} + \mu_1 p_{12} P_{n_1 +1. n_2 -1. 0} + \mu_2 p_{21} P_{n_1 -1. n_2 +1. 0} + \mu_3 P_{n_1 . n_2 . 1} \quad (34)$$
for$(n_1 < b_1 , n_2 > b_2, n_3 = 0)$

$$(\lambda_1+\lambda_2+\mu_1 +\mu_2) P_{n_1 \cdot n_2 \cdot 0} = \lambda_2 P_{n_1 . n_2 - b_2 . 0} + \mu_1 p_{12} P_{n_1 +1. n_2 -1. 0} + \mu_2 p_{21} P_{n_1 -1. n_2 +1. 0} + \mu_3 P_{n_1 . n_2 . 1} \quad (35)$$
for$(n_1 < b_1 , n_2 < b_2, n_3 = 0)$

$$(\lambda_1+\lambda_2+\mu_1) P_{n_1 \cdot 0 \cdot 0} = \mu_2 p_{21} P_{n_1 -1. 1. 0} + \mu_3 P_{n_1 . n_2 . 1} \quad (36)$$
for$(n_1 < b_1 , n_2 = 0, n_3 = 0)$

In order to solve the above system of equations (19) to (36), we apply generating function technique. For this we define generating function as:

$$F(X,Y,Z) = \sum_{n_1 =0}^{\infty} \sum_{n_2 =0}^{\infty} \sum_{n_3 =0}^{\infty} P_{n_1 \cdot n_2 \cdot n_3} X^{n_1} Y^{n_2} Z^{n_3} \quad (37)$$

Also for simplification, we define partial generating function as:

$$F_{n_2 \cdot n_3} (X) = \sum_{n_1 =0}^{\infty} P_{n_1 \cdot n_2 \cdot n_3} X^{n_1}$$
$$F_{n_3} (X, Y) = \sum_{n_2 =0}^{\infty} F_{n_2 \cdot n_3} (X) Y^{n_2} \quad (38)$$

$$F (X, Y, Z) = \sum_{n_3 =0}^{\infty} F_{n_3} (X,Y) Z^{n}{}_3 \quad (39)$$

Multiplying equation (19) & (25) by X^{n_1} and taking summation over n_1 from b_1 to ∞ and 0 to b_1 and making use of equations (31) & (38) , we obtain

$$(\lambda_1+\lambda_2+\mu_1 +\mu_2+\mu_3) F_{n_2 . n_3} (X) - \mu_1 P_{0. n_2 . n_3} = \lambda_1 X^{b_1} F_{n_2 . n_3} (X) + \lambda_2 F_{n_2 - b_2 . n_3} (X) + \frac{\mu 1 \, p_{1\,2}}{X} [F_{n_2 -1. n_3} (X) - P_{0. n_2 -1. n_3}] + \frac{\mu 1 \, p_{1\,3}}{X} [F_{n_2 . n_3 -1}(X) - P_{0. n_2 . n_3 -1}] + \mu_2 \; p_{2\,1} \; X F_{n_2 +1. n_3} (X) + \mu_2 \; p_{2\,3} \; F_{n_2 +1. n_3 -1} (X) + \mu_3 \; F_{n_2 . n_3 +1}(X) \quad (40)$$
for$(n_1 \ge b_1, n_2 > b_2, n_3 > 0)$

Multiplying equation (20) & (26) by X^{n_1} and taking summation over n_1 from b_1 to ∞ and 0 to b_1 and making use of equations (32) & (38), we obtain:

$$(\lambda_1+\lambda_2+\mu_1 +\mu_2+\mu_3)F_{n_2 \cdot n_3} (X) - \mu_1 P_{0. n_2 . n_3} = \lambda_1 X^{b_1} F_{n_2 . n_3} (X) + \frac{\mu 1 \, p_{1\,2}}{X} [F_{n_2 -1. n_3} (X) - P_{0. n_2 -1. n_3}] + \frac{\mu 1 \, p_{1\,3}}{X} [F_{n_2 . n_3 -1}(X) - P_{0. n_2 . n_3 -1}] + \mu_2 \; p_{2\,1} \; X \; F_{n_2 +1. n_3} (X) + \mu_2 \; p_{2\,3} \; F_{n_2 +1. n_3 -1}(X) + \mu_3 \; F_{n_2 . n_3 +1}(X) \quad (41)$$
for$(n_1 \ge b_1, n_2 < b_2, n_3 > 0)$

Multiplying equation (21) & (27) by X^{n_1} and taking summation over n_1 from b_1 to ∞ and 0 to b_1 and making use of equations (33) & (38), we obtain:

$$(\lambda_1+\lambda_2+\mu_1 +\mu_3)F_{0 \cdot n_3} (X) - \mu_1 P_{0.0. n_3} = \lambda_1 X^{b_1} F_{0 . n_3} (X) + \frac{\mu 1 \, p_{1\,3}}{X} [F_{0\, n_3 -1}(X) - P_{0. 0. n_3 -1}] + \mu_2 \; p_{2\,1} \; X F_{1. n_3} (X) + \mu_2 \; p_{2\,3} \; F_{1. n_3 -1} (X) + \mu_3 \; F_{0. n_3 +1}(X) \quad (42)$$
for $(n_1 \ge b_1, n_2 = 0, n_3 > 0)$

Multiplying equation (22) & (28) by X^{n_1} and taking summation over n_1 from b_1 to ∞ and 0 to b_1 and making use of equations (34) & (38), we obtain:

$$(\lambda_1+\lambda_2+\mu_1 +\mu_2)F_{n_2 . 0} (X) - \mu_1 P_{0. n_2 . 0} = \lambda_1 X^{b_1} F_{n_2 . 0} (X) + \lambda_2 F_{n_2 - b_2 . 0}(X) + \frac{\mu 1 \, p_{1\,2}}{X} [F_{n_2 -1. 0}(X) - P_{0. n_2 -1. 0}] + \mu_2 \; p_{2\,1} \; X F_{n_2 +1. 0} (X) + \mu_3 \; F_{n_2 . 1}(X) \quad (43)$$

$$\text{for}(n_1 \geq b_1, n_2 > b_2, n_3 = 0)$$

Multiplying equation (23) & (29) by X^{n_1} and taking summation over n_1 from b_1 to ∞ and 0 to b_1 and making use of equations (35) & (38), we obtain:

$$(\lambda_1+\lambda_2+\mu_1+\mu_2)F_{n_2.0}(X) - \mu_1 P_{0.n_2.0} = \lambda_1 X^{b_1} F_{n_2.0}(X) + \frac{\mu_1 p_{12}}{X}[F_{n_2-1.0}(X) - P_{0.n_2-1.0}] + \mu_2 p_{21} X F_{n_2+1.0}(X) + \mu_3 F_{n_2.0}(X) \quad (44)$$

$$\text{for}(n_1 \geq b_1, n_2 < b_2, n_3 = 0)$$

Multiplying equation (24) & (30) by X^{n_1} and taking summation over n_1 from b_1 to ∞ and 0 to b_1 and making use of equations (36) & (38), we obtain:

$$(\lambda_1+\lambda_2+\mu_1)F_{0.0}(X) - \mu_1 P_{0.0.0} = \lambda_1 X^{b_1} F_{0.0}(X) + \mu_2 p_{21} X F_{1.0}(X) + \mu_3 F_{0.1}(X) \quad (45)$$

$$\text{for}(n_1 \geq b_1, n_2 = 0, n_3 = 0)$$

Multiplying equation (40) & (41) by Y^{n_2} and taking summation over n_2 from b_2 to ∞ and 1 to b_2 -1 making use of equations (42) & (38) , we obtain

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3)F_{n_3}(X,Y) - \mu_1 F_{0.n_3}(Y) - \mu_2 F_{0.n_3}(X) &= \lambda_1 X^{b_1} F_{n_3}(X,Y) + \lambda_2 Y^{b_2} F_{n_3}(X,Y) + \frac{\mu_1 p_{12}}{X} Y[F_{n_3}(X,Y) - F_{0.n_3}(Y)] \\ &+ \frac{\mu_1 p_{13}}{X}[F_{n_3-1}(X,Y) - F_{0.n_3-1}(Y)] + \frac{\mu_2 p_{21}}{Y} X[F_{n_3}(X,Y) - F_{0.n_3}(X)] \\ &+ \frac{\mu_2 p_{23}}{Y}[F_{n_3-1}(X,Y) - F_{0.n_3-1}(X)] + \mu_3 F_{n_3+1}(X,Y) \end{aligned} \quad (46)$$

$$\text{for}(n_2 \geq b_2, n_3 > 0)$$

Multiplying equation (43) & (44) by Y^{n_2} and taking summation over n_2 from b_2 to ∞ and 1 to b_2 -1 making use of equations (45) & (38), we obtain

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2)F_0(X,Y) - \mu_1 F_{0.0}(Y) - \mu_2 F_{0.0}(X) &= \lambda_1 X^{b_1} F_0(X,Y) + \lambda_2 Y^{b_2} F_0(X,Y) + \frac{\mu_1 p_{12}}{X} Y[F_0(X,Y) - F_{0.0}(Y)] \\ &+ \frac{\mu_2 p_{21}}{Y} X[F_0(X,Y) - F_{0.0}(X)] + \mu_3 F_1(X,Y)\end{aligned} \quad (47)$$

Multiplying equation (46) by Z^{n_2} and taking summation over n_3 from 0 to ∞ and making use of equations (47) & (38, 39), we obtain

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3)F(X,Y,Z) - \mu_1 F_0(Y,Z) - \mu_2 F_0(X,Z) - \mu_3 F_0(X,Y) &= \lambda_1 X^{b_1} F(X,Y,Z) + \lambda_2 Y^{b_2} F(X,Y,Z) + \frac{\mu_1 p_{12}}{X} Y[F(X,Y,Z) - F_0(Y,Z)] \\ &+ \frac{\mu_1 p_{13}}{X} Z[F(X,Y,Z) - F_0(Y,Z)] + \frac{\mu_2 p_{21}}{Y} X[F(X,Y,Z) - F_0(X,Z)] \\ &+ \frac{\mu_2 p_{23}}{Y} Z[F(X,Y,Z) - F_0(X,Z)] + \frac{\mu_3}{Z}[F(X,Y,Z) - F_0(X,Y)]\end{aligned} \quad (48)$$

After simplifying equation (48), we get:

$$F(X,Y,Z) = \frac{\mu_1\left(1 - \frac{p_{12}Y}{X} - \frac{p_{13}Z}{X}\right)F_0(Y,Z) + \mu_2\left(1 - \frac{p_{21}X}{Y} - \frac{p_{23}Z}{Y}\right)F_0(X,Z) + \mu_3\left(1 - \frac{1}{Z}\right)F_0(X,Y)}{\lambda_1(1 - X^{b_1}) + \lambda_2(1 - Y^{b_2}) + \mu_1\left(1 - \frac{p_{12}Y}{X} - \frac{p_{13}Z}{X}\right) + \mu_2\left(1 - \frac{p_{21}X}{Y} - \frac{p_{23}Z}{Y}\right) + \mu_3\left(1 - \frac{1}{Z}\right)} \quad (49)$$

For convenience, we define

$F_0(Y,Z) = F_1$, $F_0(X,Z) = F_2$, $F_0(X,Y) = F_3$

For $X=Y=Z=1$, & $F(1,1,1) = 1$, (50)

The above equation (49) reduces to indeterminate form $\left(\frac{0}{0}\right)$.

Therefore by using L'Hospital Rule for Limits the following results are obtained from equations (49) & (50).

1) When $Y = Z = 1$, and taking $X \to 1$, we get

$$-\lambda_1 b_1 + \mu_1 - \mu_2 p_{21} = \mu_1 F_1 - \mu_2 p_{21} F_2 \quad (51)$$

2) When $X = Z = 1$, and taking $Y \to 1$, we get

$$-\lambda_2 b_2 - \mu_1 p_{12} + \mu_{21} = -\mu_1 p_{12} F_1 + \mu_2 F_2 \quad (52)$$

3) When $X = Y = 1$, and taking $Z \to 1$, we get

$$-\mu_1 p_{13} - \mu_2 p_{23} + \mu_3 = -\mu_1 p_{13} F_1 - \mu_2 p_{23} F_2 + \mu_3 F_3 \quad (53)$$

On solving these equations for F_1, F_2 & F_3, we obtain the following results -

$$F_1 = 1 - \frac{\lambda_1 b_1 + \lambda_2 b_2 p_{21}}{\mu_1 (1 - p_{12} p_{21})} = 1 - \rho_1$$

$$F_2 = 1 - \frac{\lambda_1 b_1 p_{12} + \lambda_2 b_2}{\mu_2 (1 - p_{12} p_{21})} = 1 - \rho_2$$

$$F_3 = 1 - \left\{ \frac{(\lambda_1 b_1 + \lambda_2 b_2 p_{21}) p_{13}}{\mu_3 (1 - p_{12} p_{21})} + \frac{(\lambda_1 b_1 p_{12} + \lambda_2 b_2) p_{23}}{\mu_3 (1 - p_{12} p_{21})} \right\} = 1 - \rho_3$$

Therefore,

$$P_{n_1 . n_2 . n_3} = \rho_1^{\,n_1} \rho_2^{\,n_2} \rho_3^{\,n_3} (1 - \rho_1)(1 - \rho_2)(1 - \rho_3) \quad (54)$$

Where

$$\rho_1 = \frac{\lambda_1 b_1 + \lambda_2 b_2 p_{21}}{\mu_1 (1 - p_{12} p_{21})}$$

$$\rho_2 = \frac{\lambda_1 b_1 p_{12} + \lambda_2 b_2}{\mu_2 (1 - p_{12} p_{21})}$$

$$\rho_3 = \frac{(\lambda_1 b_1 + \lambda_2 b_2 p_{21}) p_{13}}{\mu_3 (1 - p_{12} p_{21})} + \frac{(\lambda_1 b_1 p_{12} + \lambda_2 b_2) p_{23}}{\mu_3 (1 - p_{12} p_{21})} \quad (55)$$

C. QUEUE CHARACTERISTICS

From the equation (54), and (55) mean queue length can be obtained as follows:
Mean queue length (L)

$$L = L_{q_1} + L_{q_2} + L_{q_3}$$

Where

$$L_{q_1} = \frac{\rho_1}{1-\rho_1} = \frac{\lambda_1 b_1 + \lambda_2 b_2 p_{21}}{\mu_1 (1 - p_{12} p_{21}) - (\lambda_1 b_1 + \lambda_2 b_2 p_{21})}$$

$$L_{q_2} = \frac{\rho_2}{1-\rho_2} = \frac{\lambda_1 b_1 p_{12} + \lambda_2 b_2}{\mu_2 (1 - p_{12} p_{21}) - (\lambda_1 b_1 p_{12} + \lambda_2 b_2)}$$

$$L_{q_3} = \frac{\rho_3}{1-\rho_3} = \frac{(\lambda_1 b_1 + \lambda_2 b_2 P_{21}) P_{13} + (\lambda_1 b_1 P_{12} + \lambda_2 b_2) P_{23}}{\mu_3 (1 - P_{12} P_{21}) - ((\lambda_1 b_1 + \lambda_2 b_2 P_{21}) P_{13} + (\lambda_1 b_1 P_{12} + \lambda_2 b_2) P_{23})}$$

$$\therefore L = \frac{\lambda_1 b_1 + \lambda_2 b_2 p_{21}}{\mu_1 (1 - p_{12} p_{21}) - (\lambda_1 b_1 + \lambda_2 b_2 p_{21})} + \frac{\lambda_1 b_1 p_{12} + \lambda_2 b_2}{\mu_2 (1 - p_{12} p_{21}) - (\lambda_1 b_1 p_{12} + \lambda_2 b_2} + \frac{(\lambda_1 b_1 + \lambda_2 b_2 p_{21}) p_{13} + (\lambda_1 b_1 p_{12} + \lambda_2 b_2) p_{23}}{\mu_3 (1 - p_{12} p_{21}) - [(\lambda_1 b_1 + \lambda_2 b_2 p_{21}) p_{13} + (\lambda_1 b_1 p_{12} + \lambda_2 b_2) p_{23}]} \quad (56)$$

III. VALIDITY of MODEL

To check the validity of model we consider the following particular cases:-
(1) We consider
$\boldsymbol{\lambda_2 = 0, b_1 = b_2 = 1, p_{12} = 1}$ **and** $\boldsymbol{p_{21} = p_{13} = p_{23} = 0}$, from the above result (56), then we get

$$L = \frac{\lambda_1}{\mu_1 - \lambda_1} + \frac{\lambda_1}{\mu_2 - \lambda_1}$$

This coincides with that of Jackson R.R.P. [3].

(2) We consider
$\boldsymbol{b_1 = b_2 = 1, p_{13} = p_{23} = 0}$ the above result (56) becomes

$$L = \frac{\lambda_1 + \lambda_2 p_{21}}{\mu_1 (1 - p_{12} p_{21}) - (\lambda_1 + \lambda_2 p_{21})} + \frac{\lambda_1 p_{12} + \lambda_2}{\mu_2 (1 - p_{12} p_{21}) - (\lambda_1 p_{12} + \lambda_2)}$$

In that case the result resembles with Maggu [6].

(3) When we consider $\boldsymbol{p_{13} = p_{23} = 0}$ the above result (56) becomes

$$L = \frac{\lambda_1 b_1 + \lambda_2 b_2 P_{21}}{\mu_1 (1 - P_{12} P_{21}) - (\lambda_1 b_1 + \lambda_2 b_2 P_{21})} + \frac{\lambda_1 b_1 p_{12} + \lambda_2 b_2}{\mu_2 (1 - P_{12} P_{21}) - (\lambda_1 b_1 p_{12} + \lambda_2 b_2)}$$

In that case the result matches with the Hafiz Noor Mohammad et al. [9].
Thus all the above results (1), (2) & (3), shows the validity of model under consideration.

IV. NUMERICAL ILLUSTRATION

Consider the problem with the following data:-

$b_1=2$ $b_2=4$	$\lambda_1=5$ $\lambda_2=3$	$\mu_1=21$ $\mu_2=28$ $\mu_3=25$	$p_{12}=0.7$, $p_{13}=0.3$
			$p_{21}=0.4$, $p_{23}=0.6$

Find the average /mean queue length.

A. Solution Process:

In the given problem the customer enters into the system with

Batch sizes $b_1=2$, $b_2=4$,

Mean Arrival rates $\lambda_1=5$, $\lambda_2=3$,

Mean Service rates $\mu_1=21$, $\mu_2=28$, $\mu_3=25$, and

Probabilities

$p_{12}=0.7$, $p_{13}=0.3$ $p_{21}=0.4$, $p_{23}=0.6$

Such that $p_{12}+p_{13}=1$ & $p_{21}+p_{23}=1$.

Mean queue length is obtained by putting these values in the equation (56),

$$L = L_{q_1} + L_{q_2} + L_{q_3}$$

$$= \frac{\lambda_1 b_1 + \lambda_2 b_2 p_{21}}{\mu_1 (1 - p_{12} p_{21}) - (\lambda_1 b_1 + \lambda_2 b_2 p_{21})} + \frac{\lambda_1 b_1 p_{12} + \lambda_2 b_2}{\mu_2 (1 - p_{12} p_{21}) - (\lambda_1 b_1 p_{12} + \lambda_2 b_2)}$$

$$+ \frac{(\lambda_1 b_1 + \lambda_2 b_2 p_{21}) p_{13} + (\lambda_1 b_1 p_{12} + \lambda_2 b_2) p_{23}}{\mu_3 (1 - p_{12} p_{21}) - [(\lambda_1 b_1 + \lambda_2 b_2 p_{21}) p_{13} + (\lambda_1 b_1 p_{12} + \lambda_2 b_2) p_{23}]}$$

$$= \frac{14.8}{0.32} + \frac{19}{1.6} + \frac{15.84}{2.16}$$

$= 68.54$ units

V. CONCLUDING REMARKS

Most of the existing study in the field of queuing theory discussed the behavior of the model using either single arrival or batch arrival. Our model differs in the sense that we develop and design the model using the combination of both type of arrivals (i.e. single and batch) in stochastic environment with practical situation. Mean queue length of the model is obtained with the help of generating function technique. Other performance measures can be found by using Little's Formulae. Validity of the model is checked by considering the particular cases like, Jackson [3], Maggu [6] and Hafiz Noor Mohammad, Tahir Hussain & Mohammad Ikram [9]. The concept is new and one can extend the research work by including more parameters.

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