Perfect Domination in Book Graph and Stacked Book Graph

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Abstract*:*

In this paper we prove that the perfect domination number of book graph and stacked book graph are same as the domination number as the dominating set satisfies the condition for perfect domination.

Keywords:

Domination, Cartesian product graph, Perfect Domination, Book Graph, Stacked Book Graph.

I. INTRODUCTION

By a graph G = (V, E) we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by p and q respectively. For graph theoretic terminology we refer to Chartrand and Lesnaik[3]. Graphs have various special patterns like path, cycle, star, complete graph, bipartite graph, complete bipartite graph, regular graph, strongly regular graph etc. For the definitions of all such graphs we refer to Harry [7]. The study of Cross product of graph was initiated by Imrich [12]. For structure and recognition of Cross Product of graph we refer to Imrich [11]. In literature, the concept of domination in graphs was introduced by Claude Berge in 1958 and Oystein Ore in [1962] by [14]. For review of domination and its related parameters we refer to Acharya et.al. [1979] and Haynes et.al. [1998a, 1998b] [8 & 9]. The concepts of perfect domination was introduced by Cockayne et.al. [1993]. [4]. For more details we refer to Dejter and Pujol [1995], Fellows and Hoover [1991)], Yen and Lee [1996] [4 & 5]. Perfect domination is closely related to perfect codes and perfect codes have been used in coding theory. In this paper we introduce perfect domination in book graph and stacked book graph.

II. PRELIMINARIES

Definition 2.1: In graph theory, the Cartesian product $G \times H$ of graphs G and H is a graph such that

- the vertex set of $G \times H$ is the Cartesian product V(G) × V(H); and
- any two vertices (u, u') and (v, v') are adjacent in $G \times H$ if and only if either
  (i) u = v and u' is adjacent with v' in H, or
  (ii) $u' = v'$ and u is adjacent with v in G.

Definition 2.2: Book graph is a Cartesian product of a star and single edge, denoted by B_m. The m-book graph is defined as the graph Cartesian product $S_{m+1} \times P_2$, where S_{m+1} is a star graph and P_2 is the path graph . The generalization of the book graph to n "stacked" is the $(m, n) -$ Stacked book graph. We refer to [2].

Definition 2.3: The stacked book graph of order (m, n) is defined as the Cartesian product $S_{m+1} \times P_n$, where S_{m+1} is a star graph and P_n is the path graph on n nodes. It is therefore the graph corresponding to the edges of n copies of an m-page "book" stacked one on top of another and is the generalization of a book graph. we refer to [16]

Definition 2.4: A set D ⊆ V is called a dominating set if every vertex in V − D is adjacent to some vertex of D. Notice that D is a dominating set if and only if N[D] = V . The domination number of G, denoted as γ = γ(G), is the cardinality of a smallest dominating set of V . We call a smallest dominating set a γ-set.

Proposition 2.6: From [9] we know that,

1) For the Path P_n on n vertices, $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$.
2) For the Star S_n on n vertices , $\gamma(S_n) = 1$.

Definition 2.7: A subset S of V(G) is said to be a perfect dominating set if for every vertex v not in S, v is adjacent to exactly one vertex of S. Note that every perfect dominating set is a dominating set. A perfect dominating set S is said to be a minimal perfect dominating set if for any vertex v in S, S-v is not a perfect dominating set.

A perfect dominating set with smallest cardinality is called a minimum perfect dominating set. It is also called γ_{pt}- set of G. The cardinality of a γ_{pt}-set is called a perfect domination number of the graph G and is denoted as $\gamma_{pt}(G)$.

Proposition 2.9: From [4] we have results for perfect domination number of path and star graph as,

1) For the path P_n on n vertices

$$\gamma_{pt}(P_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil + 1, & \textit{where } n \textit{ is multiple of } 3 \\ \left\lceil \frac{n}{3} \right\rceil, & \textit{where } n \textit{ is not multiple of } 3 \end{cases}$$

2) For the star S_n on n vertices

$\boldsymbol{\gamma_{pt}(S_n) = 1}$ where $n \geq 2$.

III. Perfect Domination number of Book Graph and Stacked Book graph.

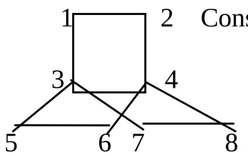
Theorem 3.1: For any book graphB_n. *we know that* $\gamma(B_n) = 2$ *where* $n \geq 3$. We refer to paper [14] for the proof.

Theorem 3.2: Perfect domination of book graph is B_n

$$\gamma_{pt}(B_n) = 2 \text{ where } n \geq 3$$

Proof: Let B_n be a book graph with $2n + 2$ vertices $V = \{v'_1, v'_2, \dots \dots \dots v'_n, v''_1, v''_2, \dots \dots \dots v''_n, v_k, v_l\}$. The book graph is the cross product of P_2 and S_{n+1}. . From the proof of $\gamma(B_n) = 2$ as given in [14], in B_n graph there are two vertices v_k and v_l which form the dominating set with the neighborhood $N(v_k) = \{v'_1, v'_2, \dots \dots \dots v'_n\}$ and $N(v_l) = \{v''_1, v''_2, \dots \dots \dots v''_n\}$. The dominating set $D = \{V_k, V_l\}$. Consider $W = V - D$. As we have N(v_k) $\cup$ $N(v_l) = V$ and $N(v_k) \cap N(v_l) = \emptyset$. Hence a vertex of W is adjacent to exactly one vertex of D, therefore D is a perfect dominating set.

Hence $\gamma_{pt}(B_n) = 2$ *where* $n \geq 3$

Example: 

Consider B_3 *dominating set* $D = \{3, 4\}$, $W = V(G) - D$

$W = \{1,2,5,6,7,8\}$

$W \cap D = \emptyset$

1, 5, 7 are adjacent to exactly one vertex 3 in D and 2, 6, 8 are adjacent to exactly one vertex 4 in the dominating set D. Hence D is a perfect dominating set of minimum cardinality. So $\gamma_{pt}(B_3) = 2$

Theorem 3.3: For any book graph B_n $\gamma[B_n(V - D)] = n$ *where* $n \geq 3$ we refer to [14]

Theorem 3.4 : For any book graph B_n

$$\gamma_{pt}[B_n(V - D)] = n \text{ where } n \geq 3$$

Proof: From Theorem 3.3 we have $\gamma[B_n(V - D)] = n$ *where* $n \geq 3$.

$B_n(V - D)$ is a disconnected graph. $B_n(V - D)$ contains n components with two vertices each say $U_1, U_2, U_3 \dots \dots \dots U_n$.

Let $U_1 = \{v'_1, v''_1\}$, $U_2 = \{v'_2, v''_2\}$, $U_3 = \{v'_3, v''_3\}$,........ ...., $U_n = \{v'_n, v''_n\}$.

Each component U_i is a path P_2. So each component has only one vertex in its dominating set. In $U_i = \{v'_i, v''_i\}$ and v'_i is dominating set *say* D_i with v''_i is adjacent to exactly one vertex of D_i. So D_i is a perfect dominating

set of $U_{i.}$ As all these components are disconnected, the dominating for $B_n(V-D)$ which is the union of n vertices is a perfect dominating set of minimum cardinality.
This show that $B_n(V-D)$ satisfied the perfect domination condition $i.e\ \gamma_{pt}[B_n(V-D)] = n \text{ where } n \geq 3$

Result 3.5: For domination of B_n and $B_n(V-D)$ we have proved that

(i) $\gamma_{pt}(\boldsymbol{B_n}) = \gamma(B_n)$
(ii) $\gamma_{pt}[B_n(V-D)] = \gamma\,[B_n(V-D)]$

Theorem 3.6: For any stacked book graph $\gamma(B_{m,n}) = n \text{ where } m \geq 3 \text{ and } n \geq 2$. We refer to paper [14] for the proof.

Theorem 3.7: For perfect domination in stacked book graph $B_{m,n}$

$$\gamma_{pt}(B_{m,n}) = n \text{ where } m \geq 3 \text{ and } n \geq 2$$

Proof: From Theorem 3.6 $\gamma(B_{m,n}) = n$. If D is the dominating set, extending similar idea as in theorem 3.2, a vertex in dominating set is adjacent to exactly m vertices of the star graph copy, so every vertex in V- D is adjacent to exactly one vertex of D, giving that D is a perfect dominating set of minimum cardinality. Hence the perfect domination number of $B_{m,n}$ is

$$\gamma_{pt}(B_{m,n}) = n \text{ where } m \geq 3 \text{ and } n \geq 2$$

We have proved following results for split domination for book and stack graph in [14].

Theorem 3.8 **:** Split domination number of book graph B_n is $\gamma^s(B_n) = 2 \text{ where } n \geq 3$

Theorem 3.9 : For book graph B_n split domination number

$$\gamma^s(B_n[(V-D)] = n \text{ where } n \geq 3$$

Theorem 3.9: Split domination in stacked book graph $B_{m,n}$ is

$$\gamma^s(B_{m,n}) = n \text{ where } m \geq 3 \text{ and } n \geq 2$$

V RESULTS

Result 1: Perfect domination in book graph and stacked book graph

Sl. No	Cross product	Dominition number	Split Domination number	Perfect Domination number
1	Book Graph B_n $(S_{m+1} \times P_2)$	2	2	2
2	$B_n(V-D)$	n	n	n
3	Stacked Book Graph $B_{m,n}$ $(S_{m+1} \times P_n)$	n	n	n

VI CONCLUSION

In the conclusion part of this paper, we can say that we have successfully described perfect domination number of book graph B_n and stacked book graph $B_{m,n}$.

We observe that minimal dominating set of book graph is also the split dominating set and perfect dominating set. Hence the domination number, split domination number and Perfect domination number of book graph are equal.

Similarly the domination number, split domination number and perfect domination number of stacked book graph are equal.

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