Some Results on General Zagreb Indices

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Abstract: *The AutographiX system conjectured* $\frac{M_1^2(G)}{n} \le \frac{M_2^1(G)}{m}$ *for the simple graph G with n vertices and m edges. We prove that the result holds for the general Zagreb indices of* l^{th} *barycentric subdivision graphs at* $\alpha \in [1,\infty)$.

Keywords: *General Zagreb Indices, Barycentric Subdivision*

1. Introduction

All graphs under discussion are finite, undirected and simple. Let G=(V,E) be a simple graph with n vertices and m edges. Denote by uv the edge of G, connecting the vertices u and v. For $u \in V(G)$, $N(u)$ denotes the set of its (first) neighbours in G. The degree of a vertex u is denoted by $d(u) = |N(u)|$. As usual $\Delta = \Delta(G)$ and $\delta = \delta(G)$ are the maximum and minimum degree of G, respectively. $P_n, K_{1,n-1}, C_n$ denotes the paths, star and cycle of n vertices.

In 1972, I.Gutman and N.Trinajstić [1,2] explored the study of total $\pi -$ electron energy on the molecular structure and introduced two vertex degree-based graph varients. These are defined as

$$M_1(G) = \sum_{v \in V(G)} d(v)^2; \quad M_2(G) = \sum_{uv \in E(G)} d(u)d(v).$$

In Chemical literature M_1 and M_2 are named as first and second Zagreb indices [3]. The Zagreb indices are used by various researchers in their QSPR and QSAR studies. These indices are also viewed as a molecular structure-descriptors [4,5]. Various results on its chemical applications, mathematical properties and developments are identified and the use of Zagreb indices were reviewed and the references are cited there in [6,7].

A spontaneous issue is raised in 2000, for comparing the values of Zagreb indices on the same graph. This leads to the AutographiX system conjecture [8]:

Conjecture 1.1. For all simple graphs G,

$$\frac{M_1^2(G)}{n} \le \frac{M_2^1(G)}{m}$$

and the bound is tight for complete graphs.

The conjecture was not true in general, various results were appeared both supporting and opposing the conjecture, for further information see the survey [9].

In 2004, A.Miličević and S.Nikolić [10] constructed the first and second variable Zagreb indices and are defined by

$$^{\alpha}M_1 = {}^{\alpha}M_1(G) = \sum_{v \in V(G)} d(v)^{2\alpha}; \qquad ^{\alpha}M_2 = {}^{\alpha}M_2(G) = \sum_{uv \in E(G)} [d(u)d(v)]^{\alpha}.$$

In 2005, X.Li and J.Zheng [11] introduced the first general Zagreb index, immediately which turns the first variable Zagreb index as a special case of it and subsequently the present authors with Gutman [12] introduced the second general Zagreb index which is same as the second variable Zagreb index and these are defined as

$$M_1^{\alpha} = M_1^{\alpha}(G) = \sum_{v \in V(G)} d(v)^{\alpha}; \qquad M_2^{\alpha} = M_2^{\alpha}(G) = \sum_{uv \in E(G)} [d(u)d(v)]^{\alpha}$$

Where $\alpha \in \mathbf{R}$ .Various properties and relations of the first general Zagreb index are discussed in [13-16].

The first and second general Zagreb indices with k-matching in G are defined as

$$M_1^{\alpha}(M) = \sum_{v \in V(M)} d(v)^{\alpha}; \qquad M_2^{\alpha}(M) = \sum_{uv \in E(M)} [d(u)d(v)]^{\alpha},$$

Where $\alpha \in \mathbf{R}$. Obviously $M_1^0(M) = 2k, M_2^0(M) = k$.

2. MAIN RESULTS

From [17], the first barycentric subdivision of a graph is the subdivision in which a new vertex is inserted in the interior of each edge. The l^{th} barycentric subdivision of a graph is the first barycentric subdivision of the $(l-1)^{th}$ barycentric subdivision, denoted by $S_l(G)$.

our goal is to provide a result supporting the Conjecture 1.1 for general Zagreb indices in case of barycentric subdivision graph of G.

Theorem 2.1. For $\alpha \in [0,1]$ and $S_l(G)$ be a l^{th} barycentric subdivision graph of G. Then,

(2.1) $$M_2^{\alpha}(S_l(G)) \geq \frac{1}{2(n+lm)} M_1^{\alpha}(S_l(G)).M_1^{\alpha+1}(S_l(G))$$

with equality holds if and only if $\alpha = 0$ or G is regular.
Proof: By the definition of second general Zagreb index,

$$M_2^{\alpha}(S_l(G)) = \frac{1}{2} \sum_{x \in V(S_l(G))} d(x)^{\alpha} \sum_{xy \in E(S_l(G))} d(y)^{\alpha}$$

By Chebyshev's sum inequality, we get

$$\left(\sum_{x \in V(S_l(G))} d(x)^{\alpha} d(x)^{0} \right) \geq \frac{1}{(n+lm)} \left(\sum_{x \in V(S_l(G))} d(x)^{\alpha} \right) \left(\sum_{x \in V(S_l(G))} d(x)^{0} \right)$$

$$M_2^{\alpha}(S_l(G)) \geq \frac{1}{2(n+lm)} \sum_{x \in V(S_l(G))} d(x)^{\alpha} \sum_{x \in V(S_l(G))} d(x)^{0} \sum_{xy \in E(S_l(G))} d(y)^{\alpha},$$

which completes our claim. Transparently equality holds for regular graphs.
For $\alpha = 1$ in (2.1), we obtain the following corollary

Corollary 2.2. Let $S_l(G)$ be a l^{th} barycentric subdivision graph of G. Then,

(2.2) $$\frac{M_1^{2}(S_l(G))}{n+lm} \leq \frac{M_2^{1}(S(_lG))}{(l+1)m},$$

with equality if and only if G is regular graph.

A. Ilić and D.Stevanović [18] proved that the conjecture holds for the first barycentric subdivision graphs, which is a special case of our corollary at l =1.

D. Vukičević [19] obtained the following inequality for variable Zagreb indices:

Theorem 2.3. For all graphs G and $\alpha \in [0, \frac{1}{2}]$, it holds that

$$\frac{M_1^{2\alpha}(G)}{n} \leq \frac{M_2^{\alpha}(G)}{m}$$

Observing the Theorem 2.3 and Corollary (2.2), immediately we obtain the following inequality:

Theorem 2.4. For $\alpha \in [0,1]$ and $S_l(G)$ be a l^{th} barycentric subdivision graph of G. Then,

(2.3) $$\frac{M_1^{2\alpha}(S_l(G))}{n+lm} \leq \frac{M_2^{\alpha}(S_l(G))}{(l+1)m}.$$

The equality holds for $\alpha = 0$ and regular graphs.

Theorem 2.5. For $\alpha \in [1,\infty)$ and $S_l(G)$ be a l^{th} barycentric subdivision graph of G. Then,

(2.4) $$\frac{M_1^{\alpha+1}(S_l(G))}{n+lm} \leq \frac{M_2^{\alpha}(S_l(G))}{(l+1)m},$$

with equality holds if and only if G is a regular with $\alpha = 1$.
Proof: There exists infinitely many classes graphs, which both satisfy or dissatisfy the inequality (2.3) for some $\alpha \in (1,\infty)$.
Case (i):
If $\frac{M_1^{2\alpha}(S_l(G))}{n+lm} \leq \frac{M_2^{\alpha}(S_l(G))}{(l+1)m}$. Definitely $\frac{M_1^{\alpha+1}(S_l(G))}{n+lm} \leq \frac{M_1^{2\alpha}(S_l(G))}{n+lm}$ at $\alpha \in [1,\infty)$
Case (ii):
If $\frac{M_1^{2\alpha}(S_l(G))}{n+lm} > \frac{M_2^{\alpha}(S_l(G))}{(l+1)m}$, Then by using Theorem (2.1), we have

$$\frac{M_2^{\alpha}(S_l(G))}{(l+1)m} \geq \left(\frac{M_1^{\alpha}(S_l(G))}{2(n+lm)}\right)\left(\frac{M_1^{\alpha+1}(S_l(G))}{n+lm}\right) \geq \left(\frac{M_1^{\alpha+1}(S_l(G))}{n+lm}\right),$$

at $\alpha \in [1,\infty)$ and equality holds for regular graphs, which completes our claim.

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