Some Properties of Unambiguously Hexagonal Tiling Recognizable Picture Languages

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ABSTRACT. In this paper we have provided different properties and devices for an equivalence theorem for unambiguously hexagonal tiling recognizable picture languages. And we got a characterization result for hexagonal picture languages computed by unambiguous hexagonal tiling systems. On the basis of this characterization, we can prove that languages in the family of unambiguous hexagonal tiling system are unambiguous.

Keywords: Unambiguous, unambiguous hexagonal tiling system, unambiguous hexagonal domino system, hexagonal picture languages.

1. Introduction

Nowadays 'two dimensional or picture languages' becomes a research field of great interest. The interest for pattern recognition and image processing motivated many researchers to do their work related to the above field. In nineties, Restivo and Giammarresi defined REC, the family of tiling recognizable picture languages [1]. REC characterizes various formalisms like different variants of tiling systems, on-line tessellation automata, Wang systems, existential monadic second order logic etc. [2], [3], [4], and [6]. If a picture language is obtained as projection of a local picture language, then it is called tiling recognizable. It is proved that the family REC is not closed under complementation [3] and it inherits many properties from the regular string language class [4].

The notion of unambiguity is an intermediate notion between determinism and non-determinism, in formal language theory. Each accepted object admits only one successful computation, in the case of an unambiguous model. Determinism is a 'local' notion, while unambiguity is a 'global' one. In [1], the authors introduced unambiguous recognizable two dimensional languages and their family is denoted by UREC.

In one-dimensional case, there exists a unique and clear notion of determinism. But in two-dimensions, such an existence is not available. A notion which can be expressed in terms of tiling systems is the notion of ambiguity. If every picture has a unique counter-image in its corresponding local language, then a tiling system is said to be unambiguous. If a two-dimensional language (picture language) is recognized by an unambiguous tiling system, then the recognizable picture language is unambiguous.

In 2003, K.S. Dersanambika et al. [9] used different formalism for hexagonal picture languages. They introduced xyz-domino systems and proved that which is equivalent to hexagonal tiling systems. In this paper we will introduce unambiguously hexagonal tiling (respectively hexagonal domino) recognizable picture languages. And we will prove properties and devices for an equivalence theorem for unambiguously hexagonal tiling recognizable picture languages.

2. Preliminaries

Here we recall the definition, structure, axis and $2 \times 2 \times 2$ Hexagonal tiles and corresponding dominos from K.S. Dersanambika et al. [9]

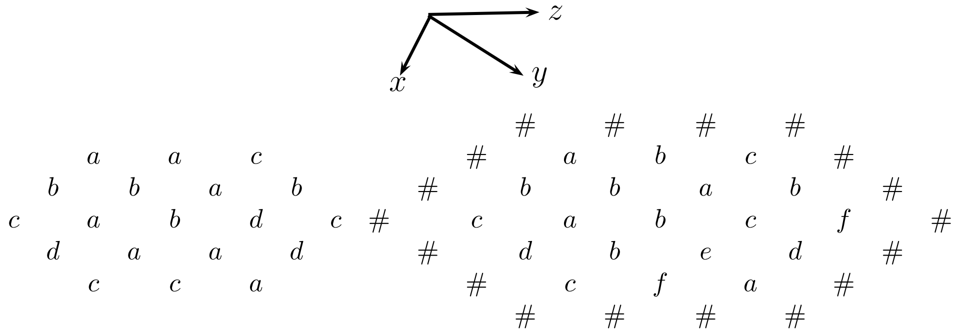


FIGURE 1

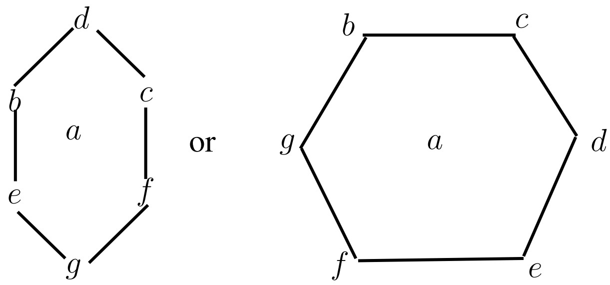


FIGURE 2

2.1. **Hexagonal Pictures.** [9] Here we recall the notions of hexagonal pictures and the hexagonal picture language [5]. Let Σ be a finite alphabet of symbols. A hexagonal picture p over Σ is a hexagonal array of symbols of Σ and the set of all hexagonal pictures over Σ is denoted by Σ^{**H}

A hexagonal picture over the alphabet $\{a, b, c, d\}$ is shown in Fig. 1.

The set of all hexagonal arrays over the alphabet Γ is denoted by Γ^{**H}.

Definition 2.1. [9] If $x \in \Gamma^{**H}$, then $\hat{x}$ is the hexagonal array obtained by surrounding x with a special boundary symbol $\# \notin \Gamma$. A hexagonal picture over the alphabet $\{a, b, c, d, e, f\}$ surrounded by $\#$ is shown in Fig. 1.

2.2. **Hexagonal Tiles and Dominos.** [9] A hexagonal picture of the form shown in Fig. 2 is called a hexagonal tile over an alphabet $\{a, b, c, d, e, f, g\}$.

Given a hexagonal picture p of size (l, m, n), for $g \leq l, h \leq m, k \leq n$ we denote $B_{g,h,k}(p)$, the set of all hexagonal blocks (or hexagonal sub-pictures) of p of size (g, h, k). $B_{2\times2\times2}$ is in fact a hexagonal tile.

Definition 2.2. [9] Let Γ be a finite alphabet. A hexagonal picture language $L \subseteq \Gamma^{**H}$ is called *local* if there exists a finite set Δ of hexagonal tiles over $\Gamma \cup \{\#\}$ such that

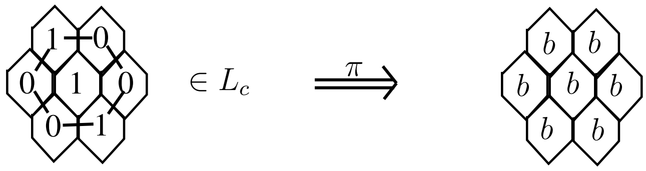
$$L = \{p \in \Gamma^{**H} / B_{2\times 2\times 2}(p) \subseteq \Delta\}$$

The family of hexagonal local picture language will be denoted by HLOC.

Definition 2.3. [9] Let Σ be a finite alphabet. A hexagonal picture language $L \subseteq \Gamma^{**H}$ is called *recognizable* if there exists a hexagonal local picture language L' (given by a set Δ of hexagonal tiles) over an alphabet Γ and a mapping $\Pi : \Gamma \to \Sigma$ such that $L = \Pi(L')$. The family of all recognizable hexagonal picture languages will be denoted by HREC.

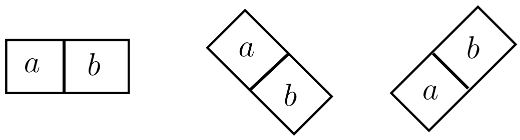
Definition 2.4. [9] A *hexagonal tiling system* is a 4-tuple $(\Sigma, \Gamma, \Pi, \theta)$ where Σ and Γ are two finite set of symbols, $\Pi : \Gamma \to \Sigma$ is a projection an θ is a set of hexagonal tiles over the alphabet $\Gamma \cup \{\#\}$.

Example 2.5. The set of all hexagons over $\Sigma = \{b\}$ is recognizable by hexagonal tiling system. Set $L' = L_c$ and $\pi(1) = \pi(0) = b$



Remark 2.6. [9] The hexagonal picture language $L \subseteq \Gamma^{**H}$ is tiling recognizable if there exists a tiling system $T = (\Sigma, \Gamma, \Pi, \theta)$ such that $L = \Pi(L(\theta))$.

Now we consider another formalism to recognize hexagonal pictures which is based on domino systems introduced by Latteux, et al. [7]. Here we consider dominos of the following types.



Definition 2.7. [9] Let L be a hexagonal picture language. The language L is xyz*-local* if there exists a set Δ of dominos as defined above over the alphabet $\Sigma \cup \{\#\}$ such that

$$L = \{\omega \in \Sigma^{**H} / \text{all domino tiles relating to } \omega \subseteq \Delta\}$$

We write $L = L(\Delta)$ if L is xyz-local and Δ is a set of dominos over Γ.

A hexagonal picture language L is xyz-domino recognizable if there exists a domino system D such that $L = \Pi(L(\Delta))$. The class of hexagonal languages recognizable by domino systems is denoted by $L(HDS)$

Theorem 2.8. [9] *If HREC is the family of all recognizable hexagonal picture languages and* $L(HDS)$ *is the class of hexagonal languages recognizable by domino systems, then we have*

$$HREC = L(HDS)$$

Lemma 2.9. [11] *Let* $\mathbb{S} : \Sigma^* \to K$ *be a rational formal series over wods. There exist rational hexagonal picture series* $S_x, S_y, S_z \in K^{rat}\langle\langle\Sigma^{**H}\rangle\rangle$ *such that for all* $P \in \Sigma^{**H}$*, we have*

$$S_x(P) = \begin{cases} S(P), & P \in \Sigma^{1\times m\times n} \\ 0, & \textit{otherwise} \end{cases}$$

$$S_y(P) = \begin{cases} S(P), & P \in \Sigma^{l\times 1\times n} \\ 0, & otherwise \end{cases}$$

$$S_z(P) = \begin{cases} S(P), & P \in \Sigma^{l\times m\times 1} \\ 0, & otherwise. \end{cases}$$

Proposition 2.10. [12] *A hexagonal picture language $\mathbb{L} \subseteq \Gamma^{**H}$ is local (respectively xyz local) if and only if its characteristic series $\mathbb{I}_{\mathbb{L}} \in \mathbb{B}\langle\langle\Sigma^{**H}\rangle\rangle$ is hexagonal tile local (xyz-local respectively).*

Proposition 2.11. [11] *Let $S \in K^{rec}\langle\langle\Sigma^{**H}\rangle\rangle$ be a series computed by a rule deterministic WHPA (weighted hexapolic picture automaton). Then S is a rational hexagonal picture series.*

Definition 2.12. [11] A weighted hexapolic picture automaton (WHPA) is an 8 tuple $\mathbb{B} = (Q, R, F_n, F_s, F_{nw}, F_{sw}, F_{ne}, F_{se})$ consisting of a finite set Q of states, a finite set of rules $R \subseteq \Sigma \times K \times Q^6$, as well as six poles of acceptance, $F_n, F_s, F_{nw}, F_{sw}, F_{ne}, F_{se} \subseteq Q$ [n-north, s-south, nw-northwest, ne-northeast, sw-southwest, se-southeast].

Proposition 2.13. [12] $K^{Ploc}\langle\langle\Sigma^{**H}\rangle\rangle \subseteq K^{rec}\langle\langle\Sigma^{**H}\rangle\rangle$.

Proposition 2.14. [11] *Let $\mathbb{B}$ be a WHPA over Σ. There exists a rule deterministic WHPA $\mathbb{B}'$ over an alphabet Γ and a projection $\Pi' : \Gamma \to \Sigma$ satisfying $\| \mathbb{B} \| = \Pi(\| \mathbb{B}' \|)$.*

3. Unambiguous hexagonal picture series

Now we extend the notion of unambiguity in the case of picture languages to that of hexagonal picture languages.

Definition 3.1. Let the alphabets be Σ and Γ and let $\mathbb{L} \subseteq \Gamma^{**H}$. Then if $\Pi' : \mathbb{L} \to \Sigma^{**H}$ is an injective mapping, then the projection $\Pi' : \Gamma \to \Sigma$ is called injective on $\mathbb{L}$.

Definition 3.2. If there exists an alphabet Γ, a hexagonal tile local (respectively xyz local) language $\mathbb{L}' \subseteq \Gamma^{**H}$ characterized by (Γ, θ') and a projection $\Pi' : \Gamma \to \Sigma$ such that Π' is injective on $\mathbb{L}'$ and $\Pi'(\mathbb{L}') = \mathbb{L}$, then a language $\mathbb{L} \subseteq \Sigma^{**H}$ is said to be an unambiguously hexagonal tiling (respectively hexagonal domino) recognizable.

The hexagonal tiling (respectively hexagonal domino) system $(\Sigma, \Gamma, \theta', \Pi')$ is called unambiguous hexagonal representation for $\mathbb{L}$.

In the case of hexagonal tiling systems, for $P \in \mathbb{L}$, $P' \in \mathbb{L}'$ satisfying $\Pi'(P') = P$ is called a pre image of P, in general. (Similar is the case with hexagonal domino systems).

UHP tile (Σ^{**H}) [respectively UHP dom (Σ^{**H})] represent the family of languages that are unambiguously hexagonal tiling (respectively hexagonal domino) recognizable over an alphabet Σ.

Now consider unambiguously hexagonal rational subsets of Σ^{**H} and the rational operations are restricted in a manner so that we will get only unambiguously hexagonal rational subsets. Let $C, D \subseteq \Sigma^{**H}$. If $C \cap D = \phi$, then only the union $C \cup D$ can be formed. The x-directional multiplication of C and D denoted by $C \varobar D$ can be formed only if $c_1 \varobar d_1 = c_2 \varobar d_2$ with $c_1, c_2 \in C$, $d_1, d_2 \in D$ implies $c_1 = c_2$, $d_1 = d_2$.

In order to form $C^{\varobar^*}$, we require that each $C^{\varobar^n}$, $n > 1$ can be formed and also the resulting sets $C, C^{\varobar^2}, C^{\varobar^3}, \ldots, C^{\varobar^n}, \ldots$ are disjoint. In a similar manner the operations $\varobslash, \oslash, \varobslash^*, \oslash^*$ are restricted. The intersection is not restricted. The above described eight operations $\cap, \cup, \varobar, \varobslash, \oslash, \varobar^*, \varobslash^*, \oslash^*$ are called unambiguously hexagonal rational operations.

Definition 3.3. The smallest class of subsets of Σ^{**H} is the class UH Rat (Σ^{**H}) of unambiguously hexagonal rational languages, denoted by $\mathbb{M}$ such that

- $\phi \in \mathbb{M}$
- all singletons are in $\mathbb{M}$
- $\mathbb{M}$ is closed under unambiguously hexagonal rational operations.

Also among the families of subsets of Σ^{**H} that contains UH Rat (Σ^{**H}), the class UHP Rat (Σ^{**H}) [unambiguously hexagonal projection of rational languages] is the smallest and which is closed under injective projections.

Note: A rational hexagonal picture expression over Σ^{**H} is called unambiguously hexagonal rational if all occuring operations are unambiguously rational.

Proposition 3.4. [10] *Every rational string language is unambiguously rational.*

Lemma 3.5. *Every xyz-local hexagonal picture language is unambiguously rational and which is a strict inclusion.*

Proof. Let $G \subseteq \Gamma^{**H}$ be xyz-local and denote G as an unambiguously rational hexagonal picture expression. Now there exist rational string languages $\mathbb{L}_x, \mathbb{L}_y, \mathbb{L}_z \subseteq \Gamma^{**H}$ with rational (string) expressions α'_x (respectively α'_y, α'_z) denoting $\mathbb{L}(\alpha'_x) = \mathbb{L}_x$ (respectively $\mathbb{L}(\alpha'_y) = \mathbb{L}_y$, $\mathbb{L}(\alpha'_z) = \mathbb{L}_z$) such that a hexagonal picture $P \in \Gamma^{**H}$ belongs to G if and only if the strings corresponding to the x-direction, y-direction or to the z-direction of p belongs to $\mathbb{L}_x, \mathbb{L}_y$ or $\mathbb{L}_z$ respectively. Clearly, the x-directional hexagonal dominos for the definition of G can be seen as the local steps for giving the definition of a local word language $\mathbb{L}_x$. Then the rational (word) expression α'_x denotes $\mathbb{L}(\alpha'_x) = \mathbb{L}_x$. Similar is the case with y-directional / z-directional hexagonal dominos. Using the preposition 3.4, assume α'_x, α'_y and α'_z to be unambiguously rational string expressions. Now let us take strings as hexagonal pictures having elements in the x-direction only (in the case of $\mathbb{L}_x$) or elements in the y-direction only (in the case of $\mathbb{L}_y$) or elements in the z-direction only (in the case of $\mathbb{L}_z$). Let the rational (hexagonal picture) expressions β'_x (respectively β'_y, β'_z) be obtained by replacing in α'_x (respectively in α'_y, α'_z) the x-directional multiplication by $\oplus$ (y-directional multiplication by $\otimes$, z-directional multiplication by $\oslash$) and $*$ operations by $\oplus^*$ (respectively $\otimes^*, \oslash^*$).

Then the unambiguity carries over from α'_x, α'_y and α'_z to β'_x, β'_y and β'_z respectively. Thus $\beta'_x, \beta'_y, \beta'_z \in$ UH Rat (Γ^{**H}) and $G = \mathbb{L}\left[(\beta'_x)^{\oplus^*} \cap (\beta'_y)^{\otimes^*} \cap (\beta'_z)^{\oslash^*}\right]$.

So we have $G \in$ UH Rat (Γ^{**H}), since the star operations $\oplus^*, \otimes^*$ and $\oslash^*$ are unambiguously hexagonal rational. To make the inclusions strict, let us consider the alphabet $\Sigma = \{0\}$. The language $\mathbb{L} = \{P \in \{0\}^{**H} / l_x(P) = 2\} \subset \Sigma^{**H}$ is clearly not xyz-local. To see $\mathbb{L} \in$ UH Rat (Σ^{**H}), denote by $\mathbb{E}'$ the unambiguously rational (word) expression for the language $\{0^n / n \geq 1\}$. Using the proof of lemma 3.1 in [11], we get an unambiguously hexagonal rational expression $\bar{\mathbb{E}}'$ for the hexagonal picture language $\{0\}^{\otimes^*}$. Then $\bar{\mathbb{E}}' \oplus \bar{\mathbb{E}}'$ and $\bar{\mathbb{E}}' \oslash \bar{\mathbb{E}}'$ are unambiguously hexagonal rational expressions for $\mathbb{L}$. ☐

Lemma 3.6. *The family UHP tile (Σ^{**H}) is closed under unambiguous x-directional / y-directional / z-directional multiplication, unambiguous x-directional / y-directional / z-directional closure, disjoint union, intersection and injective projections*

Proof. Let $\mathbb{L}_1, \mathbb{L}_2 \subseteq \Sigma^{**H}$ be unambiguously hexagonal tiling recognizable and let $(\Sigma, \Gamma_i, \theta'_i, \Pi'_i)$ $(i = 1, 2)$ be the respective unambiguous hexagonal representations. That is Π'_i is injective on $\mathcal{L}(\theta'_i), i = 1, 2$. Suppose that x-directional multiplication $\oplus$ is unambiguous for $\mathbb{L}_1$ and $\mathbb{L}_2$. Now we assume without loss of generality, the local alphabets Γ_1 and Γ_2 are disjoint. Now let us define $(\Sigma, \Gamma_1 \cup \Gamma_2, \theta', \Pi')$ for $\mathbb{L}_1 \oplus \mathbb{L}_2$ with $\mathcal{L}(\theta') = \mathcal{L}(\theta'_1) \oplus \mathcal{L}(\theta'_2)$,

$$\text{for all } b \in \Gamma_1 \cup \Gamma_2, \qquad \Pi'(b) = \begin{cases} \Pi'_1(b) & b \in \Gamma_1 \\ \Pi'_2(b) & b \in \Gamma_2 \end{cases} \tag{1}$$

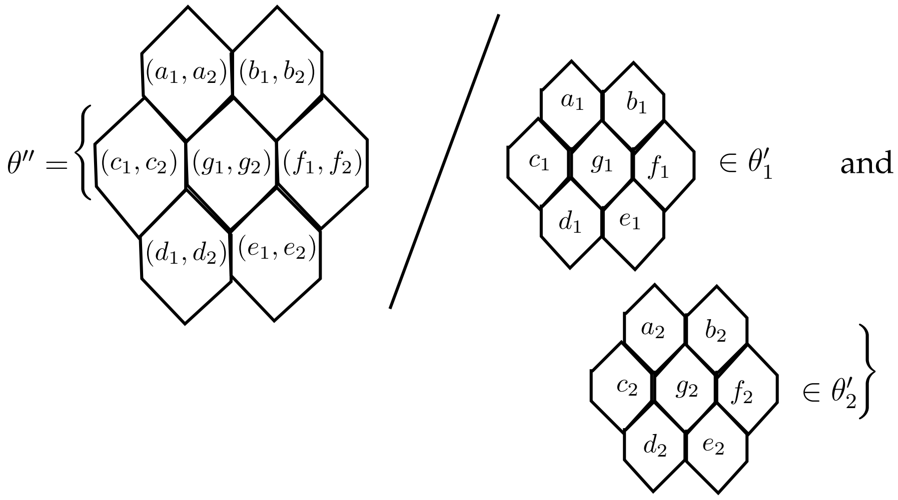
We have to prove that $\Pi' : \Gamma_1 \cup \Gamma_2 \to \Sigma$ is injective on $\mathcal{L}(\theta')$. Clearly, for $P \in \mathbb{L}_1 \oplus \mathbb{L}_2$, there exists unique $P_i \in \mathbb{L}_i$ satisfying $\Pi'_i(P'_i) = P_i$ (since $\oplus$ is unambiguous). Now we can see that, there are unique $P'_i \in \mathcal{L}(\theta'_i)$ such that $\Pi'_i(P'_i) = P_i$, hence a unique $P' \in \mathcal{L}(\theta')$ such that $\Pi'(P') = P$ (from 1)

Now by taking two disjoint unambiguous representations for $\mathbb{L}_1$ and iterating the tiles, the case of the unambiguous x-directional closure of $\mathbb{L}_1$ can be reduced to the above construction. In a similar way, we can prove the closure under unambiguous y-directional / z-directional multiplication and unambiguous y-directional / z-directional closure.

Now we have to prove the case for the disjoint union. For this, let for $i = 1, 2$, $(\Sigma, \Gamma_i, \theta'_i, \Pi'_i)$ be the respective unambiguous representation for $\mathbb{L}_i$. Suppose that $\Gamma_1 \cap \Gamma_2 = \phi$. Then $(\Sigma, \Gamma_1 \cup \Gamma_2, \theta'_1 \cup \theta'_2, \Pi')$ is a hexagonal tiling system which is an unambiguous hexagonal representation for $\mathbb{L}$.

Now we have to prove the case for intersection. For this, let $\Pi = (\Sigma, \Gamma, \theta', \Pi')$ be such that $\Gamma \subseteq \Gamma_1 \times \Gamma_2$ where $(b_1, b_2) \in \Gamma \Leftrightarrow \Pi'_1(b_1) = \Pi'_2(b_2)$.

Also, let



and from θ'', we will get θ' if we replace every $(\#, \#)$ by $\#$. And let us define $\Pi' : \Gamma \to \Sigma$ be $\Pi'((b_1, b_2)) = \Pi'_1(b_1)$ for all $(b_1, b_2) \in \Gamma$. Since we have assumed that Π'_i is injective on $\mathcal{L}(\theta')$. Therefore $(\Sigma, \Gamma, \theta', \Pi')$ is an unambiguous hexagonal system for $\mathbb{L} = \mathbb{L}_1 \cap \mathbb{L}_2$.

Let the two alphabets be Γ and Δ' and $(\Gamma, \Delta', \theta', \varphi')$ is an unambiguous hexagonal tiling system for $\mathbb{L} \subseteq \Gamma^{**H}$. Now take $\Pi' : \Gamma \to \Sigma$ as an injective projection on $\mathbb{L}$. So the hexagonal tiling system $\mathbb{T} = (\Sigma, \Delta', \theta', \varphi' \circ \Pi')$ computes $\Pi'(\mathbb{L})$. Let $P \in \Pi'(\mathbb{L})$. There exist a unique $P' \in \mathbb{L}$ with $\Pi'(P') = P$ due to the injective property of Π' on $\mathbb{L}$. Hence there exists a unique $P'' \in \mathcal{L}(\theta')$ with $\varphi'(P'') = P'$. [φ' is injective on $\mathcal{L}(\theta')$]. Therefore we got the result that $\varphi' \circ \Pi'$ is injective on $\mathcal{L}(\theta')$ and so $\mathbb{T}$ is an unambiguous hexagonal representation for $\Pi'(\mathbb{L})$. $\square$

Lemma 3.7. *We have UHP tile* $(\Sigma^{**H}) \subseteq$ *UHP dom* (Σ^{**H}). *Using the previous lemmas, we can easily prove this. If for every input hexagonal picture, there exists atmost one successful run in* $\mathbb{B}$, *then the weighted hexapolic picture automaton* $\mathbb{B}$ *is said to be unambiguous. And UH Rec* (Σ^{**H})

denote the family of languages over an alphabet Σ which is computable by unambiguous hexagonal picture automata.

Lemma 3.8. *Let $\Pi' : \Gamma \to \Sigma$ be injective on $\mathbb{L}$ and let $\mathbb{L} \in$ UH Rec (Γ^{**H}) then there exists an unambiguous hexagonal picture automaton computing $\Pi'(\mathbb{L})$*

Proof. We now convert a rule $r = (b, q_n, q_s, q_{nw}, q_{ne}, q_{sw}, q_{se})$ in an unambiguous hexagonal picture automata for $\mathbb{L}$ into a corresponding rule $r' = (\Pi'(b), q_n, q_s, q_{nw}, q_{ne}, q_{sw}, q_{se})$ in a hexagonal picture automata for $\Pi'(\mathbb{L})$. Then the resulting hexagonal picture automata is unambiguous. □

Lemma 3.9. *We have UH Rec (Σ^{**H}) = UHP tile (Σ^{**H})*

Proof. First we will prove the inclusion from right to left. For this we have to construct an unambiguous hexagonal picture automata for any local hexagonal picture language. [Since we know that languages recognized by unambiguous hexagonal picture automata are closed under injective projections] (from lemma 3.8)

Now let us assume $\mathbb{L} \subseteq \Sigma^{**H}$ to be a local hexagonal picture language. Using the preposition 2.10, we can show the $\mathbb{I}_{\mathbb{L}} \in \mathbb{B}^{\text{tile}}\langle\langle\Sigma^{**H}\rangle\rangle$ with the help of preposition 2.13, we can conclude that there exists a weighted hexapolic picture automation (WHPA) $\mathbb{B}$ computing $\mathbb{I}_{\mathbb{L}}$. Now suppose that $\mathbb{B}$ is the constructed WHPA of the proof of preposition 2.13. Here in this proof we have mentioned that $\mathbb{B}$ is unambiguous (in particular)

Let $R' \subseteq \Sigma \times \mathbb{B} \times \mathbb{Q}^6$ denote the rule set of $\mathbb{B}$. Then the hexagonal picture automata $\mathbb{A}$ with identical components to $\mathbb{B}$ with rule set

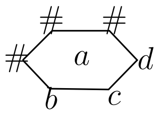
$$R'' = \{(b, q_1, q_2, q_3, q_4, q_5, q_6)/(b, 1, q_1, q_2, q_3, q_4, q_5, q_6) \in R'\}$$

satisfies $L(\mathbb{A}) = \mathbb{L}$ and clearly which is unambiguous.

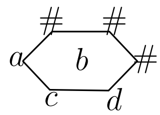
Now we have to prove the inclusion from left to right. For this let $\mathbb{L} \in$ UH Rec (Σ^{**H}). Let $\mathbb{A} = (Q, R', F_n, F_s, F_{nw}, F_{sw}, F_{ne}, F_{se})$ be an unambiguous hexagonal picture automata for $\mathbb{L}$. Because of the unambiguity of $\mathbb{A}$, the constructed projection that we have proved in the preposition 2.14 is injective on $L(\mathbb{A})$. It is enough to prove the inclusion from left to right for rule deterministic picture automata using lemma 3.6. Let $\mathbb{A} = (Q, R', F_n, F_s, F_{nw}, F_{sw}, F_{ne}, F_{se})$ be rule deterministic hexagonal picture automaton for a language $\mathbb{L}$.

For proving the remaining part, we have to use the notations in preposition 2.11 and definition 2.12.

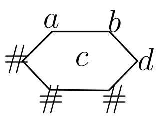
Let the arbitrary letters in Σ be a, b, c, d, e, f and g. If rules $r(a), r(b), r(c), r(d), r(e), r(f)$ and $r(g)$ are elements of R', then we define (Σ, θ') characterizing $\mathbb{L}$ as follows



$\in \theta'$ if $\sigma_{nw}(r(b)) \in F_{nw}, \sigma_{nw}(r(d)) \in F_{nw},$
$\sigma_s(r(b)) = \sigma_n(r(c)), \sigma_{ne}(r(d)) = \sigma_{sw}(r(c))$

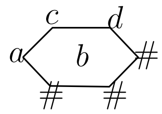


$\in \theta'$ if $\sigma_{sw}(r(a)) \in F_{sw}, \sigma_{sw}(r(d)) \in F_{sw},$
$\sigma_{se}(r(a)) = \sigma_{nw}(r(c)), \sigma_s(r(c)) = \sigma_n(r(d))$

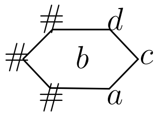


$\in \theta'$ if $\sigma_s(r(a)) = \sigma_n(r(b)), \sigma_{ne}(r(a)) \in F_{ne},$

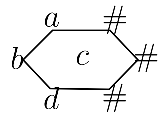
$$\sigma_{ne}(r(d)) \in F_{ne}, \sigma_{se}(r(b)) = \sigma_{nw}(r(d))$$



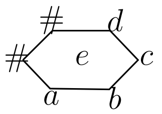
$\in \theta'$ if $\sigma_s(r(c)) = \sigma_n(r(d)), \sigma_{ne}(r(a)) = \sigma_{sw}(r(c))$
$\sigma_{se}(r(a)) \in F_{se}, \sigma_{se}(r(d)) \in F_{se}$



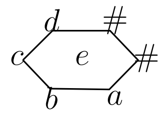
$\in \theta'$ if $\sigma_n(r(d)) \in F_n, \sigma_n(r(a)) \in F_n,$
$\sigma_{se}(r(d)) = \sigma_{nw}(r(c)), \sigma_{ne}(r(a)) = \sigma_{sw}(r(c))$



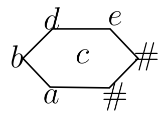
$\in \theta'$ if $\sigma_s(r(a)) \in F_s, \sigma_s(r(d)) \in F_s,$
$\sigma_{ne}(r(b)) = \sigma_{sw}(r(a)), \sigma_{se}(r(b)) = \sigma_{nw}(r(d))$



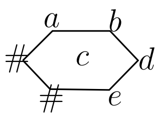
$\in \theta'$ if $\sigma_n(r(d)) \in F_n, \sigma_{nw}(r(a)) \in F_{nw}$
$\sigma_s(r(a)) = \sigma_n(r(b)), \sigma_{ne}(r(b)) = \sigma_{sw}(r(c))$
$\sigma_{se}(r(d)) = \sigma_{nw}(r(c))$



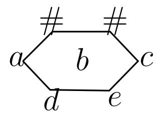
$\in \theta'$ if $\sigma_s(r(d)) \in F_s, \sigma_{sw}(r(a)) \in F_{sw}$
$\sigma_{ne}(r(c)) = \sigma_{sw}(r(d)), \sigma_{se}(r(c)) = \sigma_{nw}(r(b))$
$\sigma_s(r(b)) = \sigma_n(r(a))$



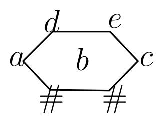
$\in \theta'$ if $\sigma_s(r(a)) \in F_s, \sigma_{se}(r(e)) \in F_{se}$
$\sigma_s(r(d)) = \sigma_n(r(e)), \sigma_{ne}(r(b)) = \sigma_{sw}(r(d))$
$\sigma_{se}(r(b)) = \sigma_{nw}(r(a))$



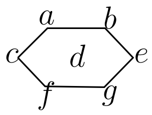
$\in \theta'$ if $\sigma_s(r(a)) = \sigma_n(r(b)), \sigma_{se}(r(b)) = \sigma_{nw}(r(d))$
$\sigma_{ne}(r(e)) = \sigma_{sw}(r(d)), \sigma_n(r(e)) \in F_n, \sigma_{ne}(r(a)) \in F_{ne}$



$\in \theta'$ if $\sigma_s(r(d)) = \sigma_n(r(e)), \sigma_{ne}(r(e)) = \sigma_{sw}(r(c))$
$\sigma_{se}(r(a)) = \sigma_{nw}(r(d)), \sigma_{sw}(r(a)) \in F_{sw}, \sigma_{nw}(r(c)) \in F_{nw}$



$\in \theta'$ if $\sigma_s(r(d)) = \sigma_n(r(e)), \sigma_{se}(r(e)) = \sigma_{nw}(r(c))$
$\sigma_{ne}(r(a)) = \sigma_{sw}(r(d)), \sigma_{se}(r(a)) \in F_{se}, \sigma_{ne}(r(c)) \in F_{ne}$



$$\in \theta' \quad \text{if } \sigma_s(r(a)) = \sigma_n(r(b)), \sigma_s(r(f)) = \sigma_n(r(g))$$
$$\sigma_{ne}(r(c)) = \sigma_{sw}(r(a)), \sigma_{ne}(r(g)) = \sigma_{sw}(r(e))$$
$$\sigma_{se}(r(c)) = \sigma_{nw}(r(f)), \sigma_{se}(r(b)) = \sigma_{nw}(r(e))$$

For a successful run $r \in R'$ on a picture $P \in \mathbb{L}$, the definition of θ' ensures that every sub hexagonal tile of P is an element of θ' and hence $P \in \mathcal{L}(\theta')$. Conversely assuming $P \in \mathcal{L}(\theta')$ we can immediately construct a successful run in $\mathbb{A}$ reading P. Then, (Σ, θ') characterizes $\mathbb{L}$ and $\mathbb{L}$ must be unambiguously hexagonal tiling recognizable since it even is hexagonal local. □

We get the following characterization result for hexagonal picture language computed by unambiguous hexagonal tiling systems.

Theorem 3.10. *Let Σ be any alphabet. We have*

$$\textit{UH Rec } (\Sigma^{**H}) = \textit{UHP Rat } (\Sigma^{**H}) = \textit{UHP tile } (\Sigma^{**H})$$
$$= \textit{UHP dom } (\Sigma^{**H})$$

Proof. Apply lemmas 3.6, 3.7, 3.8 and 3.9, we will get the above equivalence theorem.

So we can conclude that the languages in UHP tile (Σ^{**H}) are unambiguous. □

Conclusion

In this paper we introduced different devices and used various properties for the derivation of the equivalence theorem for unambiguously hexagonal tiling recognizable picture languages. In the view of different characterization, we come to a conclusion that languages in UHP tile (Σ^{**H}) are unambiguous. We can extend various properties of picture series to that of hexagonal picture series.

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