Common Fixed Point for Four Mappings in Cone Metric Space

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Abstract. *The purpose of this paper is to prove some common fixed point theorems for four mappings in cone metric space.*

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1. Introduction

In 2007, L.G Huang and X.Zhang [4] introduced cone metric space by replacing the real number set with an ordered Banach space in co-domain. After that the concept of metric space is generalised and that is cone metric space. Several authors [6, 7, 8, 9] worked on this space and proved many fixed point theorems for contractive and expansive mappings.

Recently, M Abbas and B.E Rhoades [5] obtained some fixed point theorems for pair of map in cone metric space. Later on K.Prudhvi [9] proved common fixed point theorem for three self maps in cone metric space. In this paper, we study the existence of coincidence points and then prove some fixed point theorem for four self maps in cone metric space.

2. Preliminaries

First we need the following definitions and results that will be used subsequently (see [4]).
Let E be the real Banach space with a given norm $\|\,.\,\|$ and 0 be the zero vector of E.

Definition. 2.1.

Let E be the Real Banach space with a given norm $\|\,.\,\|$ and 0 be the zero vector of E.
Then a non empty subset P of E is called a cone if and only if

1. (1) P is non-empty and $P \neq \{0\}$
2. (2) P is closed.
3. (3) $ax + by \in P$ for all $x, y \in P$ and $a, b \in R$ with $a, b \geq 0$ that is, P is convex.
4. (4) $P \cap (-P) = \{0\}$

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$ .We write $x \leq y$ to indicate $x \leq y$ but $x \neq y$ and $x << y$ will stand for $y - x \in Int(P)$.
(Int(P) = interior of P).

Definition. 2.2.

The cone $P \subset E$ is called normal if there is a number K such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$ where K is least positive number satisfying the above inequality and called normal constant of P.

Definition 2.3.

The cone $P \subset E$ is called regular if every increasing sequence which is bounded above is convergent. That is if $\{x_n\}$ is sequence such that $x_1 \leq x_2 \leq x_3 \leq \ldots\ldots\ldots\ldots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Equivalently the cone P is regular if and only if every decreasing sequence which is bounded below is convergent.

In the following, we suppose E is Banach space, P is cone in E with int $P \neq \phi$ and $\leq$ is a partial ordering with respect to P.

Definition 2.4.

Let X be non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies

(1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$.
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(3) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

then d is called a cone metric on X and (X, d) is called a cone metric space.

Example 2.5.

Let $E = R^2$, $P = \{(x, y) \in E \mid x, y \geq 0\} \subset R^2$, $X = R$ and $d : X \times X \to E$ such that $d(x, y) = (|x-y|, a|x-y|)$, where $a \geq 0$ is constant. Then (X, d) is a cone metric space.

Definition 2.6.

Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$.

(1) Then $\{x_n\}$ is said to be convergent to x if every $c \in E$ with $0 << c$ there exist N such that $d(x_n, x) << c$ for all $n \geq N$.
we denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
(2) If for every $c \in E$ with $0 << c$, there is a positive integer N such that for all $n, m > N$, $d(x_n, x_m) << c$. Then the sequence $\{x_n\}$ is called a Cauchy sequence in X.
(3) If every Cauchy sequence in X is convergent then (X, d) is called a complete cone metric space.

Lemma 2.7.

Let (X, d) be a cone metric space and P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X, then $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$

Lemma2.8.

Let (X, d) be a cone metric space and P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X, then $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$

Lemma2.9.

Let (X, d) be a cone metric space and P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X, then limit of $\{x_n\}$ is unique that is if $\{x_n\}$ is convergent to x and $\{x_n\}$ is convergent to y, then $x = y$.

Lemma2.10.

Let (X, d) be a cone metric space and P be a normal cone with normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X with $x_n \to x$ and $y_n \to y$ as $n \to \infty$.Then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.

Definition2.11[7]

Let $f, g : X \to X$ be mappings. If $w = f(z) = g(z)$ for some $z \in X$, then z is called a coincidence point of f and g and w is called a point of coincidence of f and g.

Definition2.12[6]

The mappings $f, g : X \to X$ are said to be weakly compatible if for every $x \in X$ holds

$$f(g(x)) = g(f(x)) \text{ whenever } f(x) = g(x)$$

Lemma2.13

Let X be a non-empty set and let f, g, h have a unique point of coincidence. If (f, h) and

(g, h) are weekly compatible self maps of X then f, g and h have a unique common fixed point.

Recently K. Prudhvi introduced the notion of (f-g) sequence in cone metric space.

Definition 2.14

Let (X, d) be a cone metric space and f, g, h be self mappings of X such that

$f(x) \cup g(x) \subset h(x)$ suppose $x_0 \in X$ and $x_1 \in X$ is chosen such that $hx_1 = fx_0$ and $x_2 \in X$ is chosen such that $hx_2 = gx_1$.continuing in this way, the sequence $\{hx_n\}$ such that

$y_{2n} = hx_{2n+1} = fx_{2n}$

$y_{2n+1} = hx_{2n+2} = gx_{2n+1}$, $n = 0,1,2.....$

is called a (f-g) sequence with initial point x_0.

Further K.Prudhvi proved the following results in cone metric space.

Lemma 2.15[9]

Let (X, d) be a cone matrix space, and P be a normal cone with normal constant K. Suppose that the mappings f , g and h are three self maps of X such that

$f(x) \cup g(x) \subset h(x)$ satisfying

$d(fx, gy) \leq \alpha\, d(hx, hy) + \beta\, [d(hx, fx) + d(hy, gy)] + \gamma\, [d(hx, gy) + d(hy, fx)]$

for all $x, y \in X$ where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$

Then every (f, g) sequence with initial point $x_0 \in X$ is a cauchy sequence.

Theorem 2.16[9]

Let (X, d) be cone metric space and P be a normal cone with normal constant K. Suppose f, g, h are three self maps of X such that

$f(x) \cup g(x) \subset h(x)$ satisfying

$d(fx, gy) \leq \alpha\, d(hx, hy) + \beta\, [d(hx, fx) + d(hy, gy)] + \gamma\, [d(hx, gy) + d(hy, fx)]$

for all $x, y \in X$ where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$

If f(x), g(x) or h(x) is a complete subspace of x then f, g and h have a unique point of coincidence. Moreover if (f, h) and (g, h) a weekly compatible then f, g and h have a unique common fixed point.

Now we prove these two results to obtain coincidence and common fixed point for four self- maps in cone metric space.

3.Main Results

Lemma3.1.

Let (X, d) be a cone metric space and P be a normal cone with normal constant K. Suppose A, B, S, T be self mappings of X such that $AX \subset TX$, $BX \subset SX$ satisfying

$d(Ax, By) \leq \alpha\, d(Sx, Ty) + \beta\, [d(Ax, Sx) + d(By, Ty)] + \gamma\, [d(Sx, By) + d(Ax, Ty)]$ (3.1)

for all $x, y \in X$ where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$

then for each $x_0 \in X$, the sequence $\{y_n\}$ in X is defined by the rule

$y_{2n} = Ax_{2n} = Tx_{2n+1}$

$y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$ is a Cauchy sequence. (3.2)

Proof :- Let x_0 be an arbitrary point of X and $\{y_n\}$ be a sequence used in (3.2).

Assume $Ax_n \neq Ax_{n+1}$ for all $n \in N$ then $x_n \neq x_{n+1}$ for all n.

Now using (3.1), we have

$$
\begin{aligned}
d(y_{2n}, y_{2n+1}) &= d(Tx_{2n+1}, Sx_{2n+2}) \\
&= d(Ax_{2n}, Bx_{2n+1}) \\
&\leq \alpha\, d(Sx_{2n},Tx_{2n+1}) + \beta\, [d(Ax_{2n},Sx_{2n}) + d(Bx_{2n+1},Tx_{2n+1})] \\
&\qquad + \gamma\, [d(Sx_{2n},Bx_{2n+1}) + d(Ax_{2n},Tx_{2n+1})] \\
&\leq \alpha\, d(Sx_{2n},Tx_{2n+1}) + \beta\, [d(Tx_{2n+1},Sx_{2n}) + d(Sx_{2n+2},Tx_{2n+1})] \\
&\qquad + \gamma\, [d(Sx_{2n},Sx_{2n+2}) + d(Tx_{2n+1},Tx_{2n+1})] \\
&\leq (\alpha + \beta)\, d(Sx_{2n},Tx_{2n+1}) + \beta\, d(Sx_{2n+2},Tx_{2n+1}) \\
&\qquad + \gamma\, [d(Sx_{2n},Sx_{2n+1}) + d(Sx_{2n+1},Sx_{2n+2})] \\
&\leq (\alpha + \beta)\, d(Sx_{2n},Tx_{2n+1}) + \beta\, d(Sx_{2n+2},Tx_{2n+1}) \\
&\qquad + \gamma\, [d(Sx_{2n},Tx_{2n+1}) + d(Tx_{2n+1},Sx_{2n+2})] \\
&\leq (\alpha + \beta + \gamma)\, d(Sx_{2n},Tx_{2n+1}) + (\beta + \gamma)\, d(Sx_{2n+2},Tx_{2n+1}) \\
&\leq (\alpha + \beta + \gamma)\, d(y_{2n-1}, y_{2n}) + (\beta + \gamma)\, d(y_{2n+1},y_{2n})
\end{aligned}
$$

Hence,

$$(1-(\beta + \gamma))\, d(y_{2n}, y_{2n+1}) \leq (\alpha + \beta + \gamma)\, d(y_{2n-1}, y_{2n})$$

$$d(y_{2n}, y_{2n+1}) \leq \frac{(\alpha+\beta+\gamma)}{(1-(\beta+\gamma))}\, d(y_{2n-1}, y_{2n})$$

Let $\lambda = \frac{(\alpha+\beta+\gamma)}{(1-\beta-\gamma)}$ then $\lambda < 1$.

So, $d(y_{2n}, y_{2n+1}) \leq \lambda\, d(y_{2n}, y_{2n-1})$ (3.3)

Similarly it can be shown that

$d(y_{2n+1}, y_{2n+2}) \leq \lambda\, d(y_{2n}, y_{2n+1})$

Therefore for all n,

$d(y_{n+1}, y_{n+2}) \leq \lambda\, d(y_n, y_{n+1}) \leq \ldots\ldots\ldots\ldots \leq \lambda^{n+1}\, d(y_0, y_1)$

Now for any m > n,

$d(y_n, y_m) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \dots\dots\dots\dots + d(y_{m-1}, y_m)$

$\leq (\lambda^n + \lambda^{n+1} + \dots\dots + \lambda^{m-1})\ d(y_0, y_1)$

$\leq \frac{\lambda^n}{1-\lambda} d(y_0, y_1)$

Now $\|d(y_n, y_m)\| \leq \frac{\lambda^n}{1-\lambda} \|d(y_1, y_0)\|$

Since $\lambda < 1$, $\frac{\lambda^n}{1-\lambda} \to 0$ as $n \to \infty$ which implies that $d(y_n, y_m) \to \infty$ as $n, m \to \infty$.Hence $\{y_n\}$ is a Cauchy sequence.

Theorem3.2

Let (X, d) be a cone metric space. P be a normal cone with a normal constant K. Suppose A, B, S, T be self mappings of X such that

$AX \subset TX$ and $BX \subset SX$ satisfying

$d(Ax, By) \leq \alpha\, d(Sx, Ty) + \beta\, [d(Ax, Sx) + d(By, Ty)]$

$+\gamma\, [d(Sx, By) + d(Ax, Ty)]$

For all $x, y \in X$ and $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$

If one of A(x), B(x), S(x) & T(x) is a complete subspace of X, then A, B, S,T have unique point of coincidence. Moreover, if (A, S) and (B, T) are weakly compatible, then A, B, S, T have unique common fixed point.

Proof: Let x_0 be any arbitrary point in X. Define a sequence $\{y_n\}$ in X defined by the rule

$y_{2n} = Ax_{2n} = Tx_{2n+1}$

$y_{2n+1} = Bx_{2n+1} = Sx_{sn+1}$

Now by above lemma (3.1), $\{y_n\}$ is a Cauchy sequence in X.

Now suppose Sx is a complete subspace of X. Then $y_{2n} = Sx_{2n+1}$ is also a Cauchy sequence in X. So there exist $u, v \in X$, such that $y_{2n} \to v = Su$.

Now,

$d(Su, Au) \leq d(Su, Sx_{2n}) + d(Sx_{2n}, Au)$

$\leq d(v, Sx_{2n}) + d(Au, Bx_{2n-1})$

$\leq d(v, Sx_{2n}) + \alpha\, [d(Su, Tx_{2n-1})] + \beta\, [d(Au, Su) + d(Bx_{2n-1}, Tx_{2n-1})]$

$+ \gamma\, [d(Su, Bx_{2n-1}) + d(Au, Tx_{2n-1})]$

$\leq d(v, Sx_{2n}) + \alpha\, [d(v, Tx_{2n-1})] + \beta\, [d(Au, Su) + d(Sx_{2n}, Tx_{2n-1})]$

$+ \gamma\, [d(v, Bx_{2n-1}) + d(Au, Tx_{2n-1})]$

$\leq d(v, Sx_{2n}) + \alpha\, [d(v, Sx_{2n-1})] + \beta\, [d(Au, Su) + d(Sx_{2n}, Sx_{2n-1})]$

$+ \gamma [d(v, Sx_{2n}) + d(Au, Sx_{2n-1})]$

$\leq d(v, Sx_{2n}) + \alpha [d(v, Sx_{2n-1})] + \beta [d(Au, Su) + d(Sx_{2n}, Su) + d(Su, Sx_{2n-1})]$

$+ \gamma [d(v, Sx_{2n}) + d(Au, Su) + d(Su, Sx_{2n-1})]$

$\leq d(v, Sx_{2n}) + \alpha [d(v, Sx_{2n-1})] + \beta [d(Au, Su) + d(Sx_{2n}, v) + d(v, Sx_{2n-1})]$

$+ \gamma [d(v, Sx_{2n}) + d(Au, Su) + d(v, Sx_{2n-1})]$

$\leq (1 + \beta + \gamma) d(v, Sx_{2n}) + (\alpha + \beta + \gamma) d(v, Sx_{2n-1}) + (\beta + \gamma) d(Au, Su)$

Implies, $(1- (\beta + \gamma)) d(Au, Su) \leq (1 + \beta + \gamma) d(v, Sx_{2n}) + (\alpha + \beta + \gamma) d(v,Sx_{2n-1})$

Therefore,

$d(Au, Su) \leq \frac{(1+\beta+\gamma)}{(1-\beta-\gamma)} d(v, Sx_{2n}) + \frac{(\alpha+\beta+\gamma)}{(1-\beta-\gamma)} d(v, Sx_{2n-1})$

$d(Au, Su) \leq \lambda_1 d(v, Sx_{2n}) + \lambda_2 d(v, Sx_{2n-1})$, where $\lambda_1 = \frac{(1+\beta+\gamma)}{(1-\beta-\gamma)}$ and $\lambda_2 = \frac{(\alpha+\beta+\gamma)}{(1-\beta-\gamma)}$

Taking norm on both sides,

$\|d(Au, Su)\| \leq K (\lambda_1 \|d(v, Sx_{2n})\| + \lambda_2 \|d(v, Sx_{2n-1})\|)$

Taking limit as $n \to \infty$, we have

$\|d(Au, Su)\| \to 0$ and $Au = Su = v$

Similarly by using the inequality

$d(Tu, Au) \leq d(Tu, Tx_{2n}) + d(Tx_{2n}, Au)$

we can show that $Tu = Au = v$

and also similarly using

$d(Bu, Au) \leq d(Bu, Bx_{2n}) + d(Bx_{2n}, Au)$

we can show $Bu = Au = v$

Thus $v = Au = Bu = Su = Tu$

And hence we conclude that v is coincidence point of A, B, S, T.

Now, we show that the point of coincidence is unique.

Assume that there is another point v^* that is coincidence point of A, B, S, T. So

$v^* = Au^* = Bu^* = Su^* = Tu^*$ for some $u^* \in X$

Now by (3.1),

$d(v, v^*) = d(Au, Bu^*)$

$\leq \alpha d(Su, Tu^*) + \beta [d(Au, Su) + d(Bu^*,Tu^*)] + \gamma [d(Su, Bu^*) + d(Au, Tu^*)]$

$\leq \alpha d(v, v^*) + \beta [d(v, v) + d(v^*,v^*)] + \gamma [d(v, v^*) + d(v, v^*)]$

$\leq \alpha\, d(v, v^*) + 2\gamma\, d(v,v^*)$

$\leq (\alpha + 2\gamma)\, d(v, v^*)$

Therefore ,

$(1- \alpha- 2\gamma)\, d(v, v^*) \leq 0$

Since $(1- \alpha - 2\gamma) > 0$, So $d(v, v^*) \leq 0$

Implies $v = v^*$

Since the pairs (A, S) and (B, T) are weakly compatible and by assumption v is unique point of coincidence of A, B, S, T.

Then by lemma (2.10), we get v is unique common fixed point of A, B, S, T.

Remark.

If we choose $T = I_x$ and $S = I_x$ are identity map in above theorem then we deduce the following theorem.

Theorem 3.3.

Let (X, d) be a cone metric space and P be a normal cone with normal constant K. Suppose that the mapping A and B are two self maps of X satisfying

$$d(Ax, By) \leq \alpha\, d(x, y) + \beta\, [d(Ax, x) + d(By, y)]$$

$$+ \gamma\, [d(x, By) + d(Ax, y)]$$

for all $x, y \in X$ where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$

If A(X) or B(X) is complete subspace of X, then A and B have a unique point of coincidence. Moreover if A and B are weakly compatible then A and B have a unique common fixed point.

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