Growth Properties of Composition of Two Meromorphic Functions

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Abstract

In this paper, we have proved few important results on relative growth properties of entire functions, meromorphic functions and their compositions.

Key Words : Entire function, Order of entire function, Meromorphic function, Order of meromorphic function, Relative order of entire function, Relative order of meromorphic function.

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1 Introduction

Meromorphic function is a function whose singularities are only poles in the finite plane and an entire function is a function which is analytic in the entire finite complex plane.

The maximum modulus of an entire function $f(z)$ is defined by

$$M_f(r) = sup\{|f(z)| : |z| = r\}$$

If f is non constant then $M_f(r)$ is strictly increasing and continuous function of r and the inverse function

$$M_f^{-1} : (|f(0)|, \infty) \to (0, \infty)$$

exists and $\lim\limits_{r\to\infty} M_f^{-1}(r) = \infty$

Definition 1.1 *The order of an entire function f is defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}$$

Definition 1.2 *If f and g are two entire functions then the relative order of f with respect to g is defined as*

$$\rho_g(f) = \inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \ \ for \ \ all r > r_0(\mu) > 0\}$$
$$= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}$$

Now let $h(z)$ be a non constant meromorphic function in the complex plane $\mathbb{C}$. Let us denote the number of roots of the equation $h(z) = a$ in $|z| \leq r$, with due count of multiplicity by $n(r, a)$ for any complex number a and number of poles of $h(z)$ in $|z| \leq r$ by $n(r, \infty)$ or $n(r, h)$. Let us take

$$N(r,a) = \int_0^r \frac{|n(t,a) - n(0,a)|}{t} dt + n(0,a) \log r,$$
$$N(r,h) = \int_0^r \frac{n(t,h)}{t} dt$$
$$N(r,\frac{1}{h}) = \int_0^r \frac{n(t,\frac{1}{h})}{t} dt$$
$$m(r,h) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |h(re^{i\theta})| d\theta$$

where $\log^+ x = \max\{0, \log x\}$ for all $x > 0$
Now we write

$$T_h(r) = T(r,h) = m(r,h) + N(r,h) \tag{1.1}$$

Thus we understand that $m(r, h)$ is a sort of averaged magnitude of $\log |h|$ on arcs of $|z| = r$ where $|h|$ is large. The term $N(r, h)$ relates to the number of poles. The function $T_h(r)$ is called the characteristic function of the meromorphic function $h(z)$.

2 Definitions and Lemmas

In this section we state few important definitions and important lemmas.

Definition 2.1 *The order of a meromorphic function h is defined as*

$$\rho_h = \limsup_{r\to\infty} \frac{\log T_h(r)}{\log r}$$

Definition 2.2 *The relative order of a meromorphic function h with respect to an entire function f is defined as [6]*

$$\rho_f(h) = \inf\{\lambda > 0 : T_h(r) < T_f(r^\lambda) \text{ for all } r > r_0(\lambda) > 0\}$$
$$= \limsup_{r\to\infty} \frac{\log T_f^{-1} T_h(r)}{\log r}$$

Definition 2.3 *The relative order of meromorphic function f with respect to another meromorphic function h ([1], [2]) is defined as*

$$\rho_h(f) = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log T_h(r)}$$

Lemma 2.1 *(P.18 [5]) Let g be an entire function then for all large r*

$$T_g(r) \leq \log M_g(r) \leq 3T_g(2r) \tag{2.2}$$

Lemma 2.2 *([8]) Let f and g be two entire functions. Then for a sequence of values of r tending to infinity*

$$T_{fog}(r) \geq \frac{1}{3} \log M_f(\frac{1}{8} M_g(\frac{r}{4}) + o(1)) \tag{2.3}$$

Lemma 2.3 *([4]) Let f and g be two entire functions. Then for all sufficiently large values of r*

$$M_f(\frac{1}{8} M_g(\frac{r}{2}) - |g(0)|) \leq M_{fog}(r) \leq M_f(M_g(r)) \tag{2.4}$$

Lemma 2.4 *[3] Let g be an entire function and $\alpha > 1$, $0 < \beta < \alpha$. Then for all large r*

$$M_g(\alpha r) > \beta M_g(r) \tag{2.5}$$

Lemma 2.5 *[3] Let g be an entire function with property (A). Then for any positive integer n and for all $\sigma > 1$*

$$\{M_g(r)\}^n < M_g(r^\sigma) \tag{2.6}$$

holds for all large r

Lemma 2.6 *[10] Let f be meromorphic and let g be entire and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity,*

$$T_{fog}(r) \geq T_f(\exp(r^\mu)) \tag{2.7}$$

3 Theorems and results

In this section we have obtained theorems and results which we have proved.

Theorem 3.1 *Let f, h be two entire functions of respective finite orders ρ_f, ρ_h such that $\rho_f \neq 0$ and g be a polynomial of degree m. The relative order of h with respect to fog satisfies the inequality: $\rho_{fog}(h) \geq \frac{\rho_h}{m\rho_f}$.*
The sign of equality occurs if $|g(0)| = 0$.

Proof: We know by the definition of order of entire function [3]

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}$$

Therefore for any $\epsilon > 0$ there exists $r_0(\epsilon) > 0$ such that

$$\frac{\log\log M_f(r)}{\log r} < \rho_f + \epsilon \quad \text{for all} \ \ r > r_0(\epsilon)$$

$$\text{or} \ \ M_f(r) < \exp\{r^{\rho_f+\epsilon}\} \quad \text{for all} \ \ r > r_0(\epsilon) \tag{3.8}$$

Let $\exp\{r^{\rho_f+\epsilon}\} = r_1$ *or* $r^{\rho_f+\epsilon} = \log r_1$

$$or \ \ \log r = \frac{1}{\rho_f+\epsilon}\log\log r_1 \tag{3.9}$$

This implies, $\qquad M_f(\exp\{\frac{1}{\rho_f+\epsilon}\log\log r_1\}) < r_1$

$$or \ \ M_f^{-1}(r) > exp\{\frac{1}{\rho_f+\epsilon}\log\log r\} \qquad for \ all \ r > r_0(\epsilon) \tag{3.10}$$

Also there exists a sequence $\{r_n\}$ strictly increasing and increases to ∞ such that

$$M_f(r_n) > exp\{r_n^{\rho_f-\epsilon}\} \tag{3.11}$$

Following the same steps as shown in equation(3.9) we get

$$M_f^{-1}(r_n) < exp\{\frac{1}{\rho_f-\epsilon}\log\log r_n\} \tag{3.12}$$

Similarly for the entire function h, for any $\epsilon > 0$ there exists $r_1(\epsilon) > 0$ such that

$$M_h(r) < exp\{r^{\rho_h+\epsilon}\} \qquad for \ all \ r > r_1(\epsilon) \tag{3.13}$$

and for strictly increasing sequence $\{u_n\}$, increasing to ∞

$$M_h(u_n) > exp\{u_n^{\rho_h-\epsilon}\} \tag{3.14}$$

Now let $g(z) = a_0 + a_1 z + ... + a_m z^m$. Then $M_g(r) \sim |a_m| r^m$.

Therefore for any $\epsilon > 0$ there exists $r_2(\epsilon) > 0$ such that

$$|a_m|r^m(1-\epsilon) < M_g(r) < |a_m|r^m(1+\epsilon) \qquad for\ all\ r > r_2(\epsilon) \tag{3.15}$$

Hence for all $r > r_2(\epsilon)$

$$M_g^{-1}(r) > \left\{\frac{r}{|a_m|(1+\epsilon)}\right\}^{\frac{1}{m}} \tag{3.16}$$

and

$$M_g^{-1}(r) < \left\{\frac{r}{|a_m|(1-\epsilon)}\right\}^{\frac{1}{m}} \tag{3.17}$$

By Lemma (2.3,[4]), for all sufficiently large values of r,

$$M_{fog}(r) \leq M_f(M_g(r)) \tag{3.18}$$

That implies, for all large r

$$M_g^{-1}(M_f^{-1}(r)) \leq M_{fog}^{-1}(r) \tag{3.19}$$

Now by definition of relative order of entire function with respect to another entire function, we have

$$\begin{aligned}
\rho_{fog}(h) &= \limsup_{r\to\infty} \frac{\log M_{fog}^{-1}(M_h(r))}{\log r} \\
&\geq \limsup_{r\to\infty} \frac{\log M_g^{-1}(M_f^{-1}(M_h(r)))}{\log r} \qquad [by\ equation\ (3.19)] \\
&\geq \limsup_{r\to\infty} \frac{\log M_g^{-1}(exp(\frac{1}{\rho_f+\epsilon}\log\log M_h(r)))}{\log r} \quad [by\ equation\ (3.10)] \\
&\geq \limsup_{u_n\to\infty} \frac{\log M_g^{-1}(exp(\frac{\rho_h-\epsilon}{\rho_f+\epsilon}\log u_n))}{\log u_n} \quad [by\ equation\ (3.14)] \\
&= \limsup_{u_n\to\infty} \frac{\log M_g^{-1}(u_n^{\frac{\rho_h-\epsilon}{\rho_f+\epsilon}})}{\log u_n} \\
&\geq \limsup_{u_n\to\infty} \frac{\log\left(\frac{u_n^{\frac{\rho_h-\epsilon}{\rho_f+\epsilon}}}{|a_m|(1+\epsilon)}\right)^{\frac{1}{m}}}{\log u_n} \quad [by\ equation\ (3.16)] \\
&= \limsup_{u_n\to\infty} \frac{1}{m}\left\{(\frac{\rho_h-\epsilon}{\rho_f+\epsilon})\frac{\log u_n}{\log u_n} - \frac{\log|a_m|(1+\epsilon)}{\log u_n}\right\} \\
&= \frac{1}{m}\left(\frac{\rho_h-\epsilon}{\rho_f+\epsilon}\right)
\end{aligned}$$

Since ϵ is arbitrarily small,

$$\rho_{fog}(h) \geq \frac{\rho_h}{m\rho_f} \tag{3.20}$$

By Lemma (2.3,[4]), for all sufficiently large values of r,

$$M_f(\frac{1}{8}M_g(\frac{r}{2}) - |g(0)|) \leq M_{fog}(r)$$

If $|g(0)| = 0$ then

$$M_f(\frac{1}{8}M_g(\frac{r}{2})) \leq M_{fog}(r)$$

That implies for all sufficiently large values of r,

$$M^{-1}_{fog}(r) \leq 2M^{-1}_g(8M^{-1}_f(r)) \tag{3.21}$$

Now

$$\begin{aligned}
\rho_{fog}(h) &= \limsup_{r\to\infty} \frac{\log M^{-1}_{fog}(M_h(r))}{\log r} \\
&\leq \limsup_{r\to\infty} \frac{\log(2M^{-1}_g(8M^{-1}_f(M_h(r))))}{\log r} \; [by\ equation\ (3.21)] \\
&\leq \limsup_{r_n\to\infty} \frac{\log(M^{-1}_g(8exp(\frac{1}{\rho_f-\epsilon}\log\log M_h(r_n))))}{\log r_n} \; [by\ equation\ (3.12)] \\
&\leq \limsup_{r_n\to\infty} \frac{\log(M^{-1}_g(8exp(\frac{\rho_h+\epsilon}{\rho_f-\epsilon}\log(r_n))))}{\log r_n} \; [by\ equation\ (3.13)] \\
&= \limsup_{r_n\to\infty} \frac{\log(M^{-1}_g(8(r_n)^{\frac{\rho_h+\epsilon}{\rho_f-\epsilon}}))}{\log r_n} \\
&\leq \limsup_{r_n\to\infty} \frac{\log\left(\frac{8r_n^{\frac{\rho_h+\epsilon}{\rho_f-\epsilon}}}{|a_m|(1-\epsilon)}\right)^{\frac{1}{m}}}{\log r_n} \; [by\ equation\ (3.17)] \\
&= \limsup_{r_n\to\infty} \frac{1}{m}\left\{\frac{\log 8 + \frac{\rho_h+\epsilon}{\rho_f-\epsilon}\log r_n - \log|a_m|(1-\epsilon)}{\log r_n}\right\} \\
&= \frac{1}{m}\left(\frac{\rho_h+\epsilon}{\rho_f-\epsilon}\right)
\end{aligned}$$

Since ϵ is arbitrarily small,

$$\rho_{fog}(h) \leq \frac{\rho_h}{m\rho_f} \tag{3.22}$$

So, when $|g(0)| = 0$, from equation (3.20) and equation (3.22),

$$\rho_{fog}(h) = \frac{\rho_h}{m\rho_f}$$

Theorem 3.2 *Let f, h_1 be two entire functions of respective finite orders ρ_f and ρ_{h_1} such that $\rho_f \neq 0$ and g, h_2 be two polynomials of respective degree m_1 and m_2 such that $|h_2(0)| = 0$. The relative order of h_1oh_2 with respect to fog satisfies the inequality: $\rho_{fog}(h_1oh_2) \geq \frac{m_2\rho_{h_1}}{m_1\rho_f}$.*
The sign of equality occurs if $|g(0)| = 0$.

Proof:
Let $g(z) = a_0 + a_1 z + ... + a_{m_1} z^{m_1}$ and $h_2(z) = b_0 + b_1 z + ... + b_{m_2} z^{m_2}$ be two polynomials of degree m_1 and m_2 respectively. By definition of relative order of entire function with respect to another entire function we have

$$\begin{aligned}
\rho_{fog}(h_1oh_2) &= \limsup_{r\to\infty} \frac{\log M_{fog}^{-1}(M_{h_1oh_2}(r))}{\log r} \\
&\geq \limsup_{r\to\infty} \frac{\log M_g^{-1}(M_f^{-1}(M_{h_1oh_2}(r)))}{\log r} \; [by\ equation\ (3.19)] \\
&[Since\ |h_2(0)| = 0 \text{ by assumption, using } Lemma\ (2.3),\ [4]\ we\ get] \\
&\geq \limsup_{r\to\infty} \frac{\log M_g^{-1}(M_f^{-1}(M_{h_1}(\frac{1}{8}M_{h_2}(\frac{r}{2}))))}{\log r} \\
&[Since\ h_2\ is\ a\ polynomial\ of\ degree\ m_2,\ using\ equation\ (3.15)\ we\ get] \\
&\geq \limsup_{r\to\infty} \frac{\log M_g^{-1}(M_f^{-1}(M_{h_1}(\frac{1}{8}|b_{m_2}|(1-\epsilon)(\frac{r}{2})^{m_2})))}{\log r} \\
&\geq \limsup_{u_n\to\infty} \frac{\log M_g^{-1}(M_f^{-1}(exp(\frac{1}{8}|b_{m_2}|(1-\epsilon)(\frac{u_n}{2})^{m_2})^{\rho_{h_1}-\epsilon}))}{\log u_n} \; by\ equation\ (3.14) \\
&\geq \limsup_{u_n\to\infty} \frac{\log M_g^{-1}(exp(\frac{1}{\rho_f+\epsilon} \log\log exp(\frac{1}{8}|b_{m_2}|(1-\epsilon)(\frac{u_n}{2})^{m_2})^{\rho_{h_1}-\epsilon}))}{\log u_n} \; by\ equation\ (3.10) \\
&= \limsup_{u_n\to\infty} \frac{\log M_g^{-1}(exp(\frac{1}{\rho_f+\epsilon} \log(\frac{1}{8}|b_{m_2}|(1-\epsilon)(\frac{u_n}{2})^{m_2})^{\rho_{h_1}-\epsilon}))}{\log u_n} \\
&= \limsup_{u_n\to\infty} \frac{\log M_g^{-1}(exp(\frac{\rho_{h_1}-\epsilon}{\rho_f+\epsilon} \log(\frac{1}{8}|b_{m_2}|(1-\epsilon)(\frac{u_n}{2})^{m_2})))}{\log u_n} \\
&= \limsup_{u_n\to\infty} \frac{\log M_g^{-1}\left(\frac{1}{8}|b_{m_2}|(1-\epsilon)(\frac{u_n}{2})^{m_2}\right)^{\frac{\rho_{h_1}-\epsilon}{\rho_f+\epsilon}}}{\log u_n} \\
&[Since\ g\ is\ a\ polynomial\ of\ degree\ m_1,\ \ using\ equation\ (3.16)\ we\ get]
\end{aligned}$$

$$\geq \limsup_{u_n\to\infty} \frac{\frac{1}{m_1}\log \frac{\left(\frac{1}{8}|b_{m_2}|(1-\epsilon)(\frac{u_n}{2})^{m_2}\right)^{\frac{\rho_{h_1}-\epsilon}{\rho_f+\epsilon}}}{|a_{m_1}|(1+\epsilon)}}{\log u_n}$$

$$= \limsup_{u_n\to\infty} \frac{1}{m_1} \frac{\frac{\rho_{h_1}-\epsilon}{\rho_f+\epsilon}\left(\log \frac{1}{8}|b_{m_2}|(1-\epsilon)+m_2(\log \frac{u_n}{2})\right) - \log|a_{m_1}|(1+\epsilon)}{\log u_n}$$

$$= \left(\frac{\rho_{h_1}-\epsilon}{\rho_f+\epsilon}\right)\frac{m_2}{m_1}$$

Since ϵ is arbitrarily small,

$$\rho_{fog}(h_1 o h_2) \geq \frac{m_2\rho_{h_1}}{m_1\rho_f} \tag{3.23}$$

On the other hand, using similar steps as done in Theorem (3.1), we can prove that

$$\rho_{fog}(h_1 o h_2) \leq \frac{m_2\rho_{h_1}}{m_1\rho_f} \tag{3.24}$$

Combining equation(3.23)and equation (3.24) we get $\rho_{fog}(h_1 o h_2) = \frac{m_2\rho_{h_1}}{m_1\rho_f}$

Theorem 3.3 *Let f be an entire function of finite non-zero order ρ_f, h be a meromorphic function of finite non-zero order ρ_h and g be a polynomial of degree m. The relative order of h with respect to fog satisfies the inequality $\rho_{fog}(h) \geq \frac{\rho_h}{m\rho_f}$.*
The sign of equality occurs if $|g(0)| = 0$.

Proof: From the definition of order of entire function, we have

$$\limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r} = \rho_f$$

Also for the meromorphic function h we have

$$\limsup_{r\to\infty} \frac{\log T_h(r)}{\log r} = \rho_h$$

So, for any $\epsilon > 0$ there exists $r_0(\epsilon) > 0$, $r_1(\epsilon) > 0$ such that

$$T_f(r) < r^{\rho_f+\epsilon} \; for\ all\ r > r_0(\epsilon) \tag{3.25}$$

$$T_h(r) < r^{\rho_h+\epsilon} \; for\ all\ r > r_1(\epsilon) \tag{3.26}$$

Let $r^{\rho_f+\epsilon} = r_1$. That implies $\log r = \frac{1}{\rho_f+\epsilon}\log r_1$ or $r = r_1^{\frac{1}{\rho_f+\epsilon}}$

$$Hence \qquad T_f^{-1}(r) > r^{\frac{1}{\rho_f+\epsilon}} \; for\ all\ r > r_0(\epsilon) \tag{3.27}$$

Also there exists sequence $\{r_n\}$ and $\{u_n\}$ strictly increasing and increases to ∞ such that

$$T_f(r_n) > r_n^{\rho_f - \epsilon} \tag{3.28}$$

$$and \qquad T_h(u_n) > u_n^{\rho_h - \epsilon} \tag{3.29}$$

If f and g are entire functions, then we have from [8] for all large r

$$T_{fog}(r) \le 3T_f(2M_g(r)) \tag{3.30}$$

This implies for all large r

$$T_{fog}^{-1}(r) \ge M_g^{-1}(\frac{1}{2}(T_f^{-1}(\frac{r}{3}))) \tag{3.31}$$

Let $g(z) = a_0 + a_1 + ... + a_m z^m$ be a polynomial of degree m. By definition of relative order of meromorphic function with respect to entire function [6],

$$\begin{aligned}
\rho_{fog}(h) &= \limsup_{r\to\infty} \frac{\log T_{fog}^{-1}(T_h(r))}{\log r} \\
&\ge \limsup_{r\to\infty} \frac{\log M_g^{-1}(\frac{1}{2}T_f^{-1}(\frac{T_h(r)}{3}))}{\log r} \ by\ equation\ (3.31) \\
&\ge \limsup_{u_n\to\infty} \frac{\log M_g^{-1}(\frac{1}{2}T_f^{-1}(\frac{u_n^{\rho_h-\epsilon}}{3}))}{\log u_n} \ by\ equation\ (3.29) \\
&\ge \limsup_{u_n\to\infty} \frac{\log M_g^{-1}\Big(\frac{1}{2}\big(\frac{u_n^{\rho_h-\epsilon}}{3}\big)^{\frac{1}{\rho_f+\epsilon}}\Big)}{\log u_n} \ by\ equation\ (3.27) \\
&[Since\ g \text{ is a polynomial of order m, by equation(3.16)}] \\
&\ge \limsup_{u_n\to\infty} \frac{\frac{1}{m}\Big\{\log\Big(\frac{1}{2}\big(\frac{u_n^{\rho_h-\epsilon}}{3}\big)^{\frac{1}{\rho_f+\epsilon}}\Big) - \log|a_m|(1+\epsilon)\Big\}}{\log u_n} \\
&= \limsup_{u_n\to\infty} \Big[\frac{1}{m}\Big(\frac{\rho_h-\epsilon}{\rho_f+\epsilon}\Big)\frac{\log u_n}{\log u_n} - \frac{1}{m}\frac{\log|a_m|(1+\epsilon)}{\log u_n}\Big] \\
&= \frac{1}{m}\Big(\frac{\rho_h-\epsilon}{\rho_f+\epsilon}\Big)
\end{aligned}$$

$[Since\ \epsilon > 0$ is arbitrarily small,

$$\rho_{fog}(h) \ge \frac{\rho_h}{m\rho_f} \tag{3.32}$$

Since f is entire function and g is a polynomial, by Lemma (2.3,[4]) and Lemma (2.1), if $|g(0)| = 0$, for all sufficiently large values of r,

$$M_f(\frac{1}{8}M_g\frac{r}{2}) \leq M_{fog}(r)$$
$$\text{that implies } \log(M_f(\frac{1}{8}M_g\frac{r}{2})) \leq \log(M_{fog}(r)) \leq 3T_{fog}(2r)$$
$$or\ T_{fog}^{-1}(\frac{1}{3}\log(M_f(\frac{1}{8}M_g\frac{r}{2}))) \leq 2r$$
$$\text{Let } r_1 = \frac{1}{3}\log(M_f(\frac{1}{8}M_g\frac{r}{2}))$$
$$or\ M_f(\frac{1}{8}M_g\frac{r}{2}) = \exp(3r_1)$$
$$or\ \frac{1}{8}M_g\frac{r}{2} = M_f^{-1}(\exp(3r_1))$$

$$or\ \frac{r}{2} = M_g^{-1}(8M_f^{-1}(\exp(3r_1)))$$
$$or\ r = 2M_g^{-1}(8M_f^{-1}(\exp(3r_1)))$$

Therefore for all large r,

$$T_{fog}^{-1}(r_1) \leq 4M_g^{-1}(8M_f^{-1}(\exp(3r_1))) \tag{3.33}$$

On the other hand

$$\begin{aligned}
\rho_{fog}(h) &= \limsup_{r\to\infty} \frac{\log T_{fog}^{-1}(T_h(r))}{\log r} \\
&\leq \limsup_{r\to\infty} \frac{\log(4M_g^{-1}(8M_f^{-1}(\exp(3T_h r))))}{\log r}\ [by\ equation(3.33)] \\
&\leq \limsup_{r_n\to\infty} \frac{\log\left(M_g^{-1}\left(8\left(\log(\exp(3T_h(r_n)))\right)^{\frac{1}{\rho_f-\epsilon}}\right)\right)}{\log r_n}\ [by\ equation(3.12)] \\
&= \limsup_{r_n\to\infty} \frac{\log\left(M_g^{-1}\left(8\left(3T_h(r_n)\right)^{\frac{1}{\rho_f-\epsilon}}\right)\right)}{\log r_n} \\
&\leq \limsup_{r_n\to\infty} \frac{\log\left(M_g^{-1}\left(8\left(3(r_n)^{\rho_h+\epsilon}\right)^{\frac{1}{\rho_f-\epsilon}}\right)\right)}{\log r_n}\ [by\ equation(3.26)] \\
&\leq \limsup_{r_n\to\infty} \frac{\frac{1}{m}\left[\log\left(8\left(3(r_n)^{\rho_h+\epsilon}\right)^{\frac{1}{\rho_f-\epsilon}}\right) - \log|a_m|(1-\epsilon)\right]}{\log r_n}\ [by\ equation(3.17)] \\
&= \limsup_{r_n\to\infty} \frac{\frac{1}{m}\left[\log(8.3^{\frac{1}{\rho_f-\epsilon}}) + \log\left(r_n^{\frac{\rho_h+\epsilon}{\rho_f-\epsilon}}\right)\right]}{\log r_n}
\end{aligned}$$

$$= \limsup_{r_n \to \infty} \frac{\frac{1}{m}\left[\log\left(r_n^{\frac{\rho_h+\epsilon}{\rho_f-\epsilon}}\right)\right]}{\log r_n}$$
$$= \frac{1}{m}\left(\frac{\rho_h+\epsilon}{\rho_f-\epsilon}\right)$$

Since ϵ is arbitrarily small,

$$\rho_{fog}(h) \leq \frac{\rho_h}{m\rho_f} \tag{3.34}$$

Combining equation (3.32) and equation (3.34) we get

$$\rho_{fog}(h) = \frac{\rho_h}{m\rho_f}$$

Theorem 3.4 *Let f , h be two meromorphic functions of finite non-zero orders ρ_f , ρ_h such that $\rho_f \neq 0$ and g be a polynomial of degree m. The relative order of h with respect to fog satisfies the inequality $\rho_{fog}(h) \geq \frac{\rho_h}{m\rho_f}$.*
The sign of equality occurs if $|g(0)| = 0$.

Proof: Let $g(z) = a_0 + a_1 + ... + a_m z^m$ be a polynomial of degree m. We know by ([1] ,[2])

$$\begin{aligned}
\rho_{fog}(h) &= \limsup_{r\to\infty} \frac{\log T_h(r)}{\log T_{fog}(r)} \\
&\geq \limsup_{r\to\infty} \frac{\log T_h(r)}{\log\log M_{fog}(r)} \quad [\textit{by Lemma}(2.1)] \\
&\geq \limsup_{r\to\infty} \frac{\log T_h(r)}{\log\log M_f(M_g r)} \quad [\textit{by Lemma}(2.3)] \\
&\geq \limsup_{r\to\infty} \frac{\log T_h(r)}{\log(3T_f(2M_g r))} \quad [\textit{by Lemma}(2.1)] \\
&\geq \limsup_{r\to\infty} \frac{\log T_h(r)}{\log 3 + \log(T_f(2M_g r))} \\
&= \limsup_{r\to\infty} \frac{\log T_h(r)}{\log(T_f(2M_g r))} \\
&\geq \limsup_{r\to\infty} \frac{\log T_h(r)}{\log(T_f(2|a_m|(1+\epsilon)r^m))} \quad [\textit{by equation}(3.15)] \\
&\geq \limsup_{u_n\to\infty} \frac{(\rho_h-\epsilon)\log u_n}{\log(T_f(2|a_m|(1+\epsilon)u_n^m))} \quad [\textit{by equation}(3.29)] \\
&\geq \limsup_{u_n\to\infty} \frac{(\rho_h-\epsilon)\log u_n}{(\rho_f+\epsilon)\log(2|a_m|(1+\epsilon)u_n^m)} \quad [\textit{by equation}(3.26)]
\end{aligned}$$

$$
\begin{aligned}
&= \limsup_{u_n \to \infty} \frac{(\rho_h - \epsilon)\log u_n}{(\rho_f + \epsilon)\log(2|a_m|(1+\epsilon)) + (\rho_f + \epsilon) m \log u_n} \\
&= \limsup_{u_n \to \infty} \frac{(\rho_h - \epsilon)\log u_n}{(\rho_f + \epsilon) m \log u_n} \\
&= \frac{\rho_h - \epsilon}{m\Big(\rho_f + \epsilon\Big)}
\end{aligned}
$$

Since $\epsilon > 0$ is arbitrarily small,

$$
\rho_{fog}(h) \geq \frac{\rho_h}{m\rho_f} \tag{3.35}
$$

$$
\begin{aligned}
\rho_{fog}(h) &= \limsup_{r \to \infty} \frac{\log T_h(r)}{\log T_{fog}(r)} \\
&\leq \limsup_{r \to \infty} \frac{\log T_h(r)}{\log(\frac{1}{3}\log M_{fog}(\frac{r}{2}))} \quad [by\ Lemma(2.1)] \\
&= \limsup_{r \to \infty} \frac{\log T_h(r)}{\log\log M_{fog}(\frac{r}{2})} \\
&\leq \limsup_{r \to \infty} \frac{\log T_h(r)}{\log\log M_f(\frac{1}{8}M_g(\frac{r}{4}))} \\
&[by\ Lemma(2.3),\ since\ |g(0)| = 0]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \limsup_{r \to \infty} \frac{\log T_h(r)}{\log T_f(\frac{1}{8}M_g(\frac{r}{4}))} \quad [by\ Lemma(2.1)] \\
&\leq \limsup_{r \to \infty} \frac{\log T_h(r)}{\log(T_f(\frac{1}{8}|a_m|(1+\epsilon)4^{-m}r^m))} \quad [by\ equation(3.15)] \\
&\leq \limsup_{r \to \infty} \frac{(\rho_h + \epsilon)\log r}{\log(T_f(\frac{1}{8}|a_m|(1+\epsilon)4^{-m}r^m))} \quad [by\ equation(3.26)] \\
&\leq \limsup_{u_n \to \infty} \frac{(\rho_h + \epsilon)\log u_n}{(\rho_f - \epsilon)\log(\frac{1}{8}|a_m|(1+\epsilon)4^{-m}u_n^m)} \quad [by\ equation(3.29)] \\
&= \limsup_{u_n \to \infty} \frac{(\rho_h + \epsilon)\log u_n}{(\rho_f - \epsilon) m \log u_n + (\rho_f - \epsilon)\log(\frac{1}{8}|a_m|(1+\epsilon)4^{-m})} \\
&= \frac{\rho_h + \epsilon}{m\Big(\rho_f - \epsilon\Big)}
\end{aligned}
$$

Since $\epsilon > 0$ is arbitrarily small,

$$
\rho_{fog}(h) \leq \frac{\rho_h}{m\rho_f} \tag{3.36}
$$

Combining equation (3.35) and equation (3.36) we get

$$\rho_{fog}(h) = \frac{\rho_h}{m\rho_f}$$

Theorem 3.5 *Let f, h be meromorphic functions and g be entire such that fog is meromorphic and*

(i) $\liminf_{r\to\infty} \frac{\log r}{\Big(\log T_h(r)\Big)^{\alpha}} = A$

(ii) $\liminf_{r\to\infty} \frac{\log T_f(\exp r^{\mu})}{\Big(\log r\Big)^{\beta+1}} = B$

where A and B are positive real numbers and α, β, μ are any arbitrary real numbers satisfying $0 < \alpha < 1, \beta > 0, \alpha(\beta+1) > 1$ and $0 < \mu < \rho_g \le \infty$ then $\rho_h(fog) = \infty$

Proof: By **(i)** we have for any arbitrary $\epsilon > 0$ there exists $r_0(\epsilon) > 0$ such that

$$\log r \ge (A-\epsilon)\Big(\log T_h(r)\Big)^{\alpha} \quad \text{for all } \ r > r_0(\epsilon) \tag{3.37}$$

By **(ii)** we have for any arbitrary $\epsilon > 0$ there exists $r_1(\epsilon) > 0$ such that

$$\log T_f(\exp r^{\mu}) \ge (B-\epsilon)\Big(\log r\Big)^{\beta+1} \quad \text{for all } \ r > r_1(\epsilon) \tag{3.38}$$

By definition of relative order of meromorphic function with respect to another meromorphic function given by D. Banerjee ([1], [2]) we have

$$\begin{aligned}
\rho_h(fog) &= \limsup_{r\to\infty} \frac{\log T_{fog}(r)}{\log T_h(r)} \\
&\ge \limsup_{r\to\infty} \frac{\log T_f(\exp r^{\mu})}{\log T_h(r)} \quad [by\ Lemma (2.6)] \\
&\ge \liminf_{r\to\infty} \frac{(B-\epsilon)(\log r)^{\beta+1}}{\log T_h(r)} \quad [by\ equation\ (3.38)] \\
&\ge \liminf_{r\to\infty} \frac{(B-\epsilon)(A-\epsilon)^{\beta+1}(\log T_h(r))^{\alpha(\beta+1)}}{\log T_h(r)} \quad [by\ equation (3.37)]
\end{aligned}$$

We know by Hayman [5] that $T_h(r)$ is a convex increasing function of $\log r$. Since $\alpha(\beta+1) > 1$, by the above inequality we get that for any arbitrarily small $\epsilon > 0$, A, B constants

$$\rho_h(fog) = \infty$$

Theorem 3.6 *Let f, h be meromorphic functions and g be entire such that fog is meromorphic and*

(i) $\liminf_{r\to\infty} \frac{r^{\mu}}{\Big(\log\log T_h(r)\Big)^{\alpha}} = A$

(ii) $\liminf_{r\to\infty} \frac{\log\left[\frac{\log T_f(\exp r^{\mu})}{r^{\mu}}\right]}{(r)^{\mu\beta}} = B$

where A and B are positive real numbers and α, β, μ are any arbitrary real numbers satisfying $0 < \beta < 1, \alpha > 1, \mu\beta > 1$ and $0 < \mu < \rho_g \leq \infty$ then $\rho_h(fog) = \infty$

Proof: From **(i)** we have for any arbitrary $\epsilon > 0$ there exists $r_0(\epsilon) > 0$ such that

$$r^{\mu} \geq (A-\epsilon)\Big(\log\log T_h(r)\Big)^{\alpha} \quad \text{for all} \quad r > r_0(\epsilon) \tag{3.39}$$

From **(ii)** we have for any arbitrary $\epsilon > 0$ there exists $r_1(\epsilon) > 0$ such that

$$\log\left[\frac{\log T_f(\exp r^{\mu})}{r^{\mu}}\right] \geq (B-\epsilon)(r)^{\mu\beta} \quad \text{for all} \quad r > r_1(\epsilon)$$

$$\text{That implies} \quad \frac{\log T_f(\exp r^{\mu})}{r^{\mu}} \geq \exp\Big((B-\epsilon)(r)^{\mu\beta}\Big) \tag{3.40}$$

By definition of relative order of meromorphic function with respect to another meromorphic function given by D. Banerjee ([1], [2]) we have

$$\begin{aligned}
\rho_h(fog) &= \limsup_{r\to\infty} \frac{\log T_{fog}(r)}{\log T_h(r)} \\
&\geq \limsup_{r\to\infty} \frac{\log T_f(\exp r^{\mu})}{\log T_h(r)} \quad [by\ Lemma(2.6)] \\
&= \limsup_{r\to\infty} \frac{\log T_f(\exp r^{\mu})}{r^{\mu}} . \frac{r^{\mu}}{\log T_h(r)} \\
&\geq \liminf_{r\to\infty} \exp\Big((B-\epsilon)(r)^{\mu\beta}\Big) . \frac{(A-\epsilon)\Big(\log\log T_h(r)\Big)^{\alpha}}{\log T_h(r)}
\end{aligned}$$

[By equation (3.39) and (3.40)]

We know by Hayman [5] that $T_h(r)$ is a convex increasing function of $\log r$. Since $\mu\beta, \alpha > 1$, by the above inequality we get that for any arbitrarily small $\epsilon > 0$, A, B constants

$$\rho_h(fog) = \infty$$

Theorem 3.7 *Let f and h be two meromorphic functions and g be an entire functions such that fog is meromorphic, $0 < \rho_g \leq \infty$ and $\lambda_h(f) > 0$. Then $\rho_h(fog) = \infty$*

Proof: By definition of relative order of meromorphic function with respect to another meromorphic function given by D. Banerjee ([1], [2]) we have

$$\rho_h(f) = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log T_h(r)}$$

Therefore the lower order

$$\lambda_h(f) = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log T_h(r)}$$

That implies for all arbitrary $\epsilon > 0$ there exists $r_0(\epsilon) > 0$ such that

$$\frac{\log T_f(r)}{\log T_h(r)} > \lambda_h(f) - \epsilon \qquad for\ all\ r > r_0(\epsilon)$$

Hence

$$T_f(r) > \Big(T_h(r)\Big)^{\lambda_h(f)-\epsilon} \tag{3.41}$$

By Lemma (2.6) for a sequence of values of r tending to infinity

$$\begin{aligned} T_{fog}(r) &\geq T_f(\exp r^{\mu}) \\ &\geq \Big(T_h(\exp r^{\mu})\Big)^{\lambda_h(f)-\epsilon} \quad [by\ equation(3.41) \end{aligned}$$

That implies, for a sequence of values of r tending to infinity

$$\log T_{fog}(r) \geq (\lambda_h(f) - \epsilon)\log\Big(T_h(\exp r^{\mu})\Big)$$

Therefore

$$\begin{aligned} \rho_h(fog) &= \limsup_{r\to\infty} \frac{\log T_{fog}(r)}{\log T_h(r)} \\ &\geq \limsup_{r\to\infty} \frac{\log T_f(\exp r^{\mu})}{\log T_h(r)} \quad [by\ Lemma(2.6)] \\ &\geq \liminf_{r\to\infty} \frac{(\lambda_h(f)-\epsilon)\log\Big(T_h(\exp r^{\mu})\Big)}{\log T_h(r)} \text{ [by equation(3.41)]} \end{aligned}$$

Since $T_h(r)$ is convex increasing function of $\log r$,

$$\liminf_{r\to\infty} \frac{\log\Big(T_h(\exp r^{\mu})\Big)}{\log T_h(r)} \to \infty$$

Hence $\rho_h(fog) = \infty$

Theorem 3.8 *Let f and h be two meromorphic functions and g be an entire function such that $0 < \lambda_h(g) \leq \rho_h(g) < \infty$. If for any real positive constant k*

$$\limsup_{r\to\infty} \frac{\log T_f(\alpha M_g(r))}{\log T_g(r)} = k < \infty$$

then $\lambda_h(fog) \leq k\lambda_h(g) \leq \rho_h(fog) \leq \rho_h(g)$. Where α is any real positive constant.

Proof: By definition of relative order of meromorphic function with respect to another meromorphic function given by D. Banerjee ([1], [2]) we have

$$\rho_h(fog) = \limsup_{r\to\infty} \frac{\log T_{fog}(r)}{\log T_h(r)}$$

Therefore the lower order

$$\begin{aligned}
\lambda_h(fog) &= \liminf_{r\to\infty} \frac{\log T_{fog}(r)}{\log T_h(r)} \\
&\leq \liminf_{r\to\infty} \frac{\log 3T_f(2M_g(r))}{\log T_h(r)} \quad [8] \\
&\leq \limsup_{r\to\infty} \frac{\log T_f(2M_g(r))}{\log T_g(r)} . \liminf_{r\to\infty} \frac{\log(T_g(r))}{\log T_h(r)} \\
&= k.\lambda_h(g)
\end{aligned}$$

Therefore

$$\lambda_h(fog) \leq k\lambda_h(g) \tag{3.42}$$

Again

$$\begin{aligned}
\rho_h(fog) &= \limsup_{r\to\infty} \frac{\log T_{fog}(r)}{\log T_h(r)} \\
&\geq \limsup_{r\to\infty} \frac{\log\left(\frac{1}{3}\log M_f(\frac{1}{8}M_g(\frac{r}{4}) + o(1))\right)}{\log T_h(r)} \quad [\textit{by Lemma}(2.2)] \\
&= \limsup_{r\to\infty} \frac{\log\left(\frac{1}{3}\log M_f(\frac{1}{8}M_g(\frac{r}{4}))\right)}{\log T_h(r)} \\
&\geq \limsup_{r\to\infty} \frac{\log\left(\frac{1}{3}T_f(\frac{1}{8}M_g(\frac{r}{4}))\right)}{\log T_h(r)} \quad [\textit{by Lemma } (2.1)] \\
&\geq \limsup_{r\to\infty} \frac{\log T_f(\frac{1}{8}M_g(\frac{r}{4}))}{\log T_g(r)} . \liminf_{r\to\infty} \frac{\log T_g(r)}{\log T_h(r)} \\
&= k.\lambda_h(g)
\end{aligned}$$

Therefore

$$k\lambda_h(g) \le \rho_h(fog) \tag{3.43}$$

Finally

$$\begin{aligned}
\rho_h(fog) &= \limsup_{r\to\infty} \frac{\log T_{fog}(r)}{\log T_h(r)} \\
&\le \limsup_{r\to\infty} \frac{\log 3T_f(2M_g(r))}{\log T_h(r)} \quad [by\ [8]] \\
&\le \limsup_{r\to\infty} \frac{\log T_f(2M_g(r))}{\log T_g(r)} . \limsup_{r\to\infty} \frac{\log T_g(r)}{\log T_h(r)} \\
&= k\rho_h(g)
\end{aligned}$$

Hence

$$\rho_h(fog) \le k\rho_h(g) \tag{3.44}$$

From equation (3.42), equation(3.43), and (3.44) the theorem follows.

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