A Technical Note in awe of Nonstandard Analysis

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Abstract

Nonstandard analysis is a branch of Mathematics introduced by Abraham Robinson[1] in 1966. In 1977, Edward Nelson[4] gave an axiomatic approach to Non-standard analysis. In many instances, analysis on infinite sets can be reduced to a finiteness argument using Nonstandard methods. In this expository article, we present an introduction to the theory and atleast twice make an observation that numbers within our reach are "finitely many".

Keywords : *Nonstandard analysis, Continuous function.*

I. INTRODUCTION

Abraham Robinson[1] constructed a superstructure to work in any given structure like the Euclidean spaces, topological spaces, algebraic structures (rings, fields etc., .) , graphs and so on. Instead, Edward Nelson[4] restructured the axiomatics of set theory by introducing three new principles (IST) - Idealization, Standardization, Transfer - to the Zermelo Fraenkel set of axioms with the axiom of choice (ZFC) . Nelson proved the consistency of the new system (IST + ZFC) . This allows standard and nonstandard elements to work within sets. First we present the axioms and discuss the immediate outcomes.

II. AXIOMATICS

Henceforth whatever we refer to as 'classical' is anything we have come across in Mathematics so far. For instance, sets, cartesian products of sets, relations and functions studied so far, all axioms, mathematical structures and results in classical set theory (ZFC) still hold in our extended analysis - namely Nonstandard analysis. Like the binary predicate $'\in'$ (belongs to) and its governing rules in classical set theory, [4]Nelson introduces a unary predicate (for example, complement operation in sets is a unary predicate in classical set theory) 'standard' and spells out its governing rules in the following three axioms. We present the axioms with some discussions in between. The fundamentals are as in Alain Robert [3].

Idealization (I) : Let R (x, y) be a classical relation between two sets X and Y, that is, $R \subseteq X \times Y$. If for every standard and finite $F \subseteq X$, there exists $y_F \in Y$ such that $R(x, y_F)\ \forall x \in F$, then there exists $y \in Y$ such that R (x, y) for all standard $x \in X$.

We use the symbols $\forall^s$, $\exists^s$, $\forall^f$, $\exists^f$, $\forall^{sf}$ and $\exists^{sf}$ to mean 'for every standard', 'there exists standard', 'for every finite', 'there exists finite', 'for every standard and finite' and 'there exists standard and finite' respectively. Hence restated, Idealization axiom is as follows :

Let R (x, y) be a classical relation between two sets X and Y. That is, $R \subseteq X \times Y$. $\forall^{sf}\ F \subseteq X, \exists\ y_F \in Y$ such that $R(x, y_F)\ \forall x \in F \Rightarrow \exists\ y \in Y$ such that $R(x, y)\ \forall^s\ x \in X$.

We discuss some consequences before proceeding to the next two axioms. An element which is not standard will be called nonstandard.

Consequence 1. *Every infinite set has nonstandard elements.*

Proof. Let X be an infinite set and R be a relation on X defined by R (x, y) if $x \neq y$. Now $\forall^{sf} F \subseteq X$, $\exists\, y_F \in X$ such that R (x, y_F) , since X is infinite. By (I), $\exists\, y \in X$ such that R (x, y) $\forall^{s}\, x \in X$. That is, $y \neq x$ $\forall^{s}\, x \in X$. This y must be nonstandard. //

Consequence 2. Given any set X, there exists a finite subset F of X containing all standard elements of X.

Proof. Let P_f (X) denote the collection of all finite subsets of X. Let R be a relation between X and P_f (X) defined by R (x, A) if $x \in A$. $\forall^{sf} A \subseteq X$, $\exists\, A \in P_f(X)$ such that R (x, A) $\forall\, x \in A$. By (I), $\exists\, F \in P_f(X)$ such that R (x, F) $\forall^{s}\, x \in X$. That is, $x \in F$ $\forall^{s}\, x \in X$. Hence proved. //

Thus there is a finite subset of R containing all standard real numbers. By Transfer axiom to be spelt out, the real numbers that we know are all standard and hence are contained in a finite set. This may be rephrased as **"The real numbers within our reach are finitely many (!)".**

Next we move to Standardization axiom.

Standardization (S) : Let P be a property (classical or not) on a standard set X. Then there exists a unique standard subset E of X such that the standard elements of E are precisely the standard elements of X satisfying the property P. E is denoted by

$^{S}\{x \in X / P(x)\}$.

It will be appropriate to discuss consequences of Standardization axiom after presenting Transfer axiom and some of its consequences.

Transfer (T) : Let F be a formula involving a variable x and standard parameters A,B,C etc.,. Then

$(\forall^{s} x)\,[F(x, A, B, C...)] \Leftrightarrow (\forall x)\,[F(x, A, B, C...)]$ (*). Applying it for the negation $\neg F$ of F, we get

$(\forall^{s} x)\,[\neg F(x, A, B, C...)] \Leftrightarrow (\forall x)\,[\neg F(x, A, B, C...)]$. This is equivalent to the dual Transfer Principle.

$(\exists^{s} x)\,[F(x, A, B, C...)] \Leftrightarrow (\exists x)\,[F(x, A, B, C...)]$ (**).

Thus the Transfer principle is valid with the existential quantifier $'\exists'$ in place of the universal quantifier $'\forall'$ too.

Consequence 3. The Transfer may be extended to any finite number of quantifiers. For instance, if A,B,C..... are standard parameters, $(\forall x)(\forall y)\,.....\,[F(x, y, ...A, B....)] \Leftrightarrow (\forall^{s} x)(\forall^{s} y)\,....\,[F(x, y, ...A, B....)]$.

Consequence 4. In view of (**), whenever an entity is uniquely established in classical theory, this entity must be standard. Thus the numbers like 3, $-101/103$, e, π, $\sqrt{2}$, the set of natural numbers N, the set of real numbers R etc.,. are all standard.

Consequence 5. Let A,B be standard sets. To show $A \subseteq B$, it is enough to check $x \in A \Rightarrow x \in B$ for standard elements. This follows from Transfer axiom: $(\forall^{s} x)\,[x \in A \Rightarrow x \in B] \Leftrightarrow (\forall x)\,[x \in A \Rightarrow x \in B]$. Thus two standard sets are equal if both have the same standard elements.

Consequence 6. In a set E every element is standard if and only if E is a finite and standard set.

Proof.

$(\exists\, x \in E)$ [x is nonstandard]

$\Leftrightarrow (\exists\, x \in E)(\forall^{s}\, y \in E)\,[y \neq x]$

$\Leftrightarrow (\forall^{sf} F)\,(\exists\, x \in E)(\forall\, y \in F)\,[y \neq x]$, by (I).

$\Leftrightarrow (\forall^{sf} F)\,(\exists\, x \in E)\,[x \notin F]$

$\Leftrightarrow (\forall^{sf} F)\ [E \nsubseteq F]$

Negating the above equivalence, $(\forall x \in E)$ [x is standard] $\Leftrightarrow(\exists^{sf} F)\ [E \subseteq F]$ (*)

We shall use the above equivalence (*) to establish the statement of consequence 6.

If E is standard and finite, taking F = E, the implication $\Leftarrow$ of (*) gives that every $x \in E$ is standard.

Conversely let every $x \in E$ be standard. Then $\Rightarrow$ of (*) gives a standard, finite F such that $E \subseteq F$. First of all this says E is finite. F is standard and finite implies the power set P (F) is standard and finite. (P (F) is standard, by Consequence 4). By what we have established, every element of P (F) is standard and hence E is standard. This completes the proof. //

From classical analysis, we know that $f : R \to R$ defined by $f(x)$ =1 if x is rational

= 0 if x is irrational

is discontinuous at all real points. As a surprise we prove that there is a continuous $f : R \to R$ taking the value 1 at all rationals in our reach and 0 at all irrationals in our reach. Precisely we spell it out as an

Observation : There is a continuous $f: R \to R$ such that $f(x)$ = 1 if x is standard rational

= 0 if x is standard irrational

Proof : Let F be a finite subset of R containing all standard real numbers.

Let $F = \{x_1, x_2, \dots\dots.., x_m, y_1, y_2, \dots\dots.. y_n\}$ where x_i's are rationals and y_j's are irrationals.

Consider $f(x) = \frac{(x-x_2)(x-x_3).....(x-x_m)(x-y_1)......(x-y_n)}{(x_1-x_2)(x_1-x_3).....(x_1-x_m)(x_1-y_1).....(x_1-y_n)}$

$+ \frac{(x-x_1)(x-x_3).....(x-x_m)(x-y_1)......(x-y_n)}{(x_2-x_1)(x_2-x_3).....(x_2-x_m)(x_2-y_1).....(x_2-y_n)}$

+..

$+ \frac{(x-x_1)(x-x_2).....(x-x_{m-1})(x-y_1)......(x-y_n)}{(x_m-x_1)(x_m-x_2).....(x_m-x_{m-1})(x_m-y_1).....(x_m-y_n)}$

Clearly f is continuous (in fact, it is a polynomial function).

Also $f(x_i)$ = 0 for i = 1,2.....,m and $f(y_i)$ = 0 for j = 1,2,.....n.

Hence f is the required function. //

ACKNOWLEDEGMENT

The first author thanks Department of Science and Technology (DST), India, for its financial support through INSPIRE fellowship.

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