(G,D)-Bondage and (G,D)-Nonbondage Number of a Graph

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Abstract: *The (G,D)-bondage number of a graph G denoted by* $b\gamma_G(G)$ *is the least positive integer k such that there exists* $F \subseteq E(G)$ *with* $|F| = k$ *and* $\gamma_G(G-F) > \gamma_G(G)$. *If no such k exists, it is defined to be* ∞. *The (G,D)-nonbondage number of a graph G denoted by* $b_n\gamma_G(G)$ *is defined as the maximum cardinality among all sets of edges* $X \subseteq E(G)$ *such that* $\gamma_G(G-X) = \gamma_G(G)$. *If* $b_n\gamma_G(G)$ *does not exist, we define* $b_n\gamma_G(G) = 0$. *In this paper we initiate a study of these two parameters.*

Keywords: *Domination, Geodomination, (G, D)-number, (G, D)-bondage number and (G, D)-nonbondage number.*

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1. Introduction: Throughout this paper, we consider G as a finite undirected graph with no loops and multiple edges. The study of domination in graphs was begun by Ore and Berge[6]. Let G = (V, E) be any graph. A dominating set of a graph G is a set D of vertices of G such that every vertex in V–D is adjacent to atleast one vertex in D and the minimum cardinality among all dominating sets is called the domination number γ(G). The concept of geodominating (or geodetic) set was introduced by Buckley and Harary in [1] and Chartrand, Zhang and Harary in [2, 3, 4]. Let u, v ∈V(G). A u-v geodesic is a u-v path of length d(u, v). A vertex x ∈V(G) is said to lie on a u-v geodesic P if x is a vertex of P including the vertices u and v. A set S of vertices of G is a geodominating(or geodetic) set if every vertex of G lies on an x-y geodesic for some x, y in S. The minimum cardinality of a geodominating set is the geodomination(or geodetic) number of G and is denoted as g(G)[1, 2, 3, 4]. A (G, D)-set of G is a subset S of V(G) which is both a dominating and geodetic set of G. A (G, D)-set S of G is said to be a minimal (G, D)-set of G if no proper subset of S is a (G, D)-set of G. The minimum cardinality of all (G, D)-sets of G is called the (G, D)-number of G and it is denoted by γ_G(G). Any (G, D)-set of G of cardinality γ_G is called a γ_G-set of G [8, 9, 10].

Fink et al. [5] introduced the bondage number of a graph in 1990. The bondage number $b(G)$ of a graph G is the cardinality of a smallest set of edges whose removal from G results in a graph with domination number greater than $\gamma(G)$.

In [7], Kulli and Janakiram introduced the concept of the nonbondage number as follows: The nonbondage number $b_n(G)$ of G is the maximum cardinality of all sets of edges $X \subseteq E$ for which $\gamma(G-X) = \gamma(G)$ for an edge set X, then X is called the nonbondage set and the maximum one the maximum nonbondage set. If $b_n(G)$ does not exist, we define $b_n(G) = 0$.

Let G = (V, E) be any graph and v ∈V(G). The neighbourhood of v, written as N_G(v) or N(v) is defined by N(v) = {x∈ V(G) : x is adjacent to v}. The *degree* of a vertex v in a graph G is defined to be the number of edgesincident with v and is denoted by $degv$. A vertex of degree zero is an isolated vertex and a vertex of degree one is a *pendant vertex* (or end vertex). Any vertex which is adjacent to a pendant vertex is called a support. A graph G is complete if every pair of distinct vertices of $G$are adjacent in G. A complete graph on p vertices denoted by K_p.A graph G is called *acyclic* if it has no cycles. A connected acyclic graph is called a *tree*.

***Remark*1.1:[8]** $\left\lceil \frac{n-4}{3} \right\rceil + 2 = \begin{cases} \left\lceil \frac{n}{3} \right\rceil & if n \equiv 1 (mod\ 3) \\ \left\lceil \frac{n}{3} \right\rceil + 1 & otherwise. \end{cases}$

Theorem1.2:[8]

$$\gamma_G(P_n) = \begin{cases} \left\lceil \frac{n-4}{3} \right\rceil + 2 & if n \geq 5 \\ 2 & if n = 2, 3 \text{ or } 4. \end{cases}$$

Proposition 1.3:[8] For $n > 5, \gamma_G(C_n) = \gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

Theorem 1.4:[8] Let $W_p = C_{p-1} + K_1, p \geq 5$ denote the wheel graph on p vertices. Then, $\gamma_G(W_p) = \left\lfloor \frac{p}{2} \right\rfloor$.

Corollary 1.5:[8] Let $G = (V, E)$ be a connected graph on p vertices. Then, $\gamma_G(G) = p$ if and only if G is complete.

Notation 1.6:$K_n(m_1, m_2, \dots, m_n)$ denotes the graph obtained from K_n by pasting $m_1, m_2, \dots, m_n$ edges to the vertices $u_1, u_2, \dots, u_n$ of K_n respectively.

2. (G,D)-BONDAGE NUMBER OF A GRAPH

Definition 2.1: The (G,D)-bondage number of a graph G denoted by b$\gamma_G(G)$ is the least positive integer k such that there exists F ⊆ E(G) with $|F| = k$ and $\gamma_G(G - F) > \gamma_G(G)$. If no such k exists, it is defined to be ∞.

Remark 2.2: (i) If $\gamma_G(G) = p$, then b$\gamma_G(G)$=∞. Hence, b$\gamma_G(K_p)$=∞. (ii) (G,D)-number is defined for connected graphs with atleast two vertices [8]. So, let us assume that (G,D)-number of a disconnected graph is the sum of (G,D)-number of its components. (iii) Also, assume that (G,D)-number of a graph with less than two vertices, that is, graph is a single vertex is 1.

Proposition 2.3: b$\gamma_G(P_n) = 1$ for all n ≥ 3.

Proof: Obviously, b$\gamma_G(P_3)$ = b$\gamma_G(P_4)$ = b$\gamma_G(P_5)$ = b$\gamma_G(P_6)$ =1.Let n ≥ 7. Remove an edge e from P_n such that $P_n - \{e\} = P_5 \cup P_{n-5}$.

Then, $\gamma_G(P_n - \{e\}) = \gamma_G(P_5) + \gamma_G(P_{n-5})$.

***Case* 1:** $n = 7, 8$ or 9

$$\gamma_G(P_n - \{e\}) = \gamma_G(P_5) + \gamma_G(P_{n-5})$$

$= 3 + 2$(by theorem 1.2)

$$= 5$$

But, $\gamma_G(P_7) = 3$ and $\gamma_G(P_8) = \gamma_G(P_9) = 4$.

Therefore, $\gamma_G(P_n - \{e\}) = 5 > \gamma_G(P_n)$.

Hence, b$\gamma_G(P_n) = 1$.

Case 2: $n \geq 10$

$$\gamma_G(P_n - \{e\}) = \gamma_G(P_5) + \gamma_G(P_{n-5})$$

$= 3 + \left\lceil \frac{n-5-4}{3} \right\rceil + 2$(by theorem 1.2)

$$= 3 + \left\lceil \frac{n-9}{3} \right\rceil + 2.$$

By theorem 1.2, $\gamma_G(P_n) = \left\lceil \frac{n-4}{3} \right\rceil + 2$.

Now, $3 + \left\lceil \frac{n-9}{3} \right\rceil = \left\lceil \frac{9}{3} \right\rceil + \left\lceil \frac{n-9}{3} \right\rceil \geq \left\lceil \frac{n}{3} \right\rceil > \left\lceil \frac{n-4}{3} \right\rceil$.

Therefore, $\gamma_G(P_n - \{e\}) = 3 + \left\lceil \frac{n-9}{3} \right\rceil + 2$

$$> \left\lceil \frac{n-4}{3} \right\rceil + 2$$

$= \gamma_G(P_n)$.

Hence, b$\gamma_G(P_n) = 1$ for all n ≥ 3.

Proposition 2.4: For n > 5,

b$\gamma_G(C_n) = \begin{cases} 1 & if n \equiv 0 (mod\ 3) or n \equiv 2 (mod\ 3) \\ 2 & otherwise. \end{cases}$

Proof:

Case 1: $n \equiv 0\ or\ 2\ (mod\ 3)$

Then, $\gamma_G(C_n - \{e\}) = \gamma_G(P_n)$

$= \left\lceil \frac{n}{3} \right\rceil + 1$(by theorem 1.2 & remark 1.1)

$> \left\lceil \frac{n}{3} \right\rceil = \gamma_G(C_n)$(by proposition 1.3)

Therefore, b$\gamma_G(C_n) = 1$.

Case 2: $n \equiv 1\ (mod\ 3)$

Then, $\gamma_G(C_n - \{e\}) = \gamma_G(P_n)$

$= \left\lceil \frac{n}{3} \right\rceil$(by theorem 1.2 & remark 1.1)

$= \gamma_G(C_n)$(by proposition 1.3).

But, b$\gamma_G(P_n) = 1$(by proposition 2.3).

Therefore, b$\gamma_G(C_n) = 2$.

Proposition 2.5: b$\gamma_G(K_{1,n}) = n - 1$.

Proof: $\gamma_G(K_{1,n}) = n$ and it becomes $n + 1$ only if we remove $n - 1$ of its n edges. That is, $\gamma_G(K_{1,n} - F) = n + 1$ only when $|F| = n - 1$. Therefore, b$\gamma_G(K_{1,n}) = |F| = n - 1$.

Proposition 2.6: b$\gamma_G(W_p) = \begin{cases} 1 & if p is odd \\ 2 & otherwise. \end{cases}$

Proof: We know that, $W_p = C_{p-1} + K_1$.

By theorem (1.4), $\gamma_G(W_p) = \left\lfloor \frac{p}{2} \right\rfloor$.

When p is odd, $p - 1$ is even. Now, if we remove any one edge from the cycle C_{p-1}, then any set containing $\left\lfloor \frac{p}{2} \right\rfloor$ vertices of W_p is not a (G,D)-set. Therefore, $\gamma_G(W_p - \{e\}) = \left\lfloor \frac{p}{2} \right\rfloor + 1 > \gamma_G(W_p)$. Hence, b$\gamma_G(W_p) = 1$ if p is odd.

If p is even, then only by removing atleast two consecutive edges from C_{p-1}, we get the γ_G-value increased. Therefore, b$\gamma_G(W_p) = 2$ if p is even.

Proposition 2.7: If $\gamma_G(G) \neq p$, then $1 \leq b\gamma_G(G) \leq e$, where e is the cardinality of the edge set of G.

***Proof*:** Obviously, b$\gamma_G(G) \geq 1$. By removing all the edges from G, obviously $G - E$ is a totally disconnected graph and so $\gamma_G(G - E) = p > \gamma_G(G)$. Therefore, b$\gamma_G(G) \leq |E| = e$. Hence, $1 \leq b\gamma_G(G) \leq e$.

Theorem 2.8: Let T be any tree with $k \geq 2$ support vertices and l end vertices such that $l + k = p$. Let L and K be the set of all end and support vertices of T respectively. Then, b$\gamma_G(T) = min_{v\epsilon K}\{|N(v) \cap L|\}$.

Proof: Let v be a support vertex of T such that $|N(v) \cap L|$ is minimum. Assume that v is adjacent to h end vertices in T.

Case 1: v is adjacent to exactly one support vertex

Remove all the edges incident with the end vertices adjacent to v. In the resultant graph $T^{'}$, v is an end vertex. Any minimum (G, D)-set of T together with v forms a minimum (G, D)- set of $T^{'}$. Therefore, $\gamma_G(T^{'}) = \gamma_G(T) + 1 > \gamma_G(T)$. Obviously, $h = |N(v) \cap L|$. Also, removal of no set of less than h edges, increases the (G, D)- number of the resulting graph. Therefore, b$\gamma_G(T) = min_{v\epsilon K}\{|N(v) \cap L|\}$.

Case 2: v is adjacent to atleast two support vertices

Let $T^{''}$ be the graph obtained by removing all the edges incident with the end vertices of v. In $T^{''}$, no minimum (G, D)- set of T dominates v. Further, any minimum (G, D)-set of T together with v forms a minimum (G, D)-set of $T^{''}$. Therefore, $\gamma_G(T^{''}) = \gamma_G(T) + 1 > \gamma_G(T)$. As before, $h = |N(v) \cap L|$ and removal of no set of less than h edges, increases the (G, D)-number of the resulting graph.

Therefore, $b\gamma_G(T) = min_{v\epsilon K}\{|N(v) \cap L|\}$.

Theorem 2.9: For every positive integer $k \geq 1$, there exists a graph G with $b\gamma_G(G) = k$.

***Proof*:** Let $k \geq 1$. Consider $G \cong K_n(k, 0, 0, \dots)$ and $|V(G)| = n + k = p$. Let $V(K_n) = \{v_1, v_2, \dots, v_n\}$. Then, G is obtained by pasting k edges to v_1 say, $u_i v_1 (i = 1, 2, \dots, k)$. Clearly, $V(G) - \{v_1\}$ is the unique minimum (G, D)- set of G and so, $\gamma_G(G) = n - 1 + k = p - 1$. Obviously, removal of any set of $t < k$ edges from G, does not increase the (G, D)- number of G. Let $G^{'}$ be the graph obtained by removing the k edges pasted to v_1 from G. Then, $G^{'} \cong K_n \cup [K_1 \cup K_1 \cup \dots (k times)]$ and so, $\gamma_G(G^{'}) = n + k = p > \gamma_G(G)$. Also, $\{u_i v_1 : 1 \leq i \leq k\}$ is the unique bγ_G-set of G. Again, removal of no set of less than k edges increases the (G, D)- number of the resulting graph. Therfore, $b\gamma_G(G) = k$.

Proposition 2.10:

b$\gamma_G(G_\mathbf{1} \cup G_\mathbf{2}) = \min\{b\gamma_G(G_1), b\gamma_G(G_2)\}$, for any two graphs G_1 and G_2.

Proof: Let b$\gamma_G(G_1) = m(\leq q_1)$ and b$\gamma_G(G_2) = n(\leq q_2)$ Where, $|E(G_1)| = q_1$ and $|E(G_2)| = q_2$. Then, removal of m edges from G_1 increases the γ_G- value of G_1 and removal of n edges from G_2 increases the γ_G-value of G_2. Take $\gamma_G(G_1) = r$ and $\gamma_G(G_2) = t$. So, $\gamma_G(G_1 \cup G_2) = r + t$. Thus, removal of m edges from G_1, $\gamma_G(G_1 \cup G_2)$ is strictly greater than $r + t$. Also removal of n edges from G_2, $\gamma_G(G_1 \cup G_2)$ is strictly greater than $r + t$. Therefore, b$\gamma_G(G_1 \cup G_2) = \min\{m, n\} = \min\{b\gamma_G(G_1)$, b$\gamma_G(G_2)\}$.

3. (G,D)-NONBONDAGE NUMBER OF A GRAPH

Definition 3.1: The (G,D)-nonbondage number of a graph G denoted by $b_n\gamma_G(G)$ is defined as the maximum cardinality among all sets of edges $X \subseteq E(G)$ such that $\gamma_G(G - X) = \gamma_G(G)$.

Proposition3.2: For $n > 5$,

$b_n\gamma_G(C_n) = \begin{cases} 1 & if n \equiv 1(mod\ 3) \\ 0 & otherwise. \end{cases}$

Proof:

Case 1: $n \equiv 1(mod\ 3)$

Removal of any one edge from C_n results in P_n. Further, $\gamma_G(C_n) = \gamma_G(P_n)$ if and only if $n = 3k + 1$. That is, $n \equiv 1(mod\ 3)$. Therefore, $b_n\gamma_G(C_n) \geq 1$. Further, removal of more edges from C_n results in a disconnected graph with atleast two components. And, sum of their γ_G-values is greater than $\gamma_G(C_n)$. Therefore, $b_n\gamma_G(C_n) = 1$.

Case 2: $n \equiv 0(mod\ 3)$ or $n \equiv 2(mod\ 3)$

$$\gamma_G(C_n - \{e\}) = \gamma_G(P_n)$$

$= \left\lceil \frac{n}{3} \right\rceil + 1$(by theorem 1.2 & remark 1.1)

$= \gamma_G(C_n) + 1$(by proposition 1.3)

$> \gamma_G(C_n)$.

Therefore, $b_n\gamma_G(C_n) = 0$.

Proposition 3.3: $b_n\gamma_G(K_{1,n}) = n - 2$.

Proof: In a star graph, even if we remove $n - 2$ of the n edges incident with the central vertex, γ_G-value is not changed and is equal to n. But, if we remove $n - 1$ of the n edges incident with the central vertex, the resultant graph is $[K_1 \cup K_1 \cup \dots (n-1)times] \cup K_2$. So,

$$\gamma_G(K_{1,n} - \{e_1, e_2, \dots, e_{n-1}\}) = n - 1 + \gamma_G(K_2)$$

$$= n - 1 + 2$$

$$= n + 1$$

$> \gamma_G(K_{1,n})$.

Therefore, $b_n\gamma_G(K_{1,n}) = n - 2$.

Proposition 3.4: Let G be a connected graph on p vertices. Then, $b_n\gamma_G(G) = q$if and only if G is complete, where $q = |E(G)|$.

Proof: Let G be a graph on p vertices. Suppose $b_n\gamma_G(G) = q$. Then, $\gamma_G(G) = \gamma_G(G - E(G))$ ……………(1). Since $q = |E(G)|$, $G - E(G)$ is the totally disconnected graph on p vertices and $\gamma_G(G - E(G)) = p$. Therefore, $\gamma_G(G) = p$ [by equation (1)]. By corollary 1.5, G is complete. Conversely, suppose G is complete. Then, by corollary 1.5, $\gamma_G(G) = p$. Since G is complete, $\gamma_G(G - E(G)) = p = \gamma_G(G)$.Therefore, $b_n\gamma_G(G) = q$.

Objective 3.5:$0 \leq \gamma_G(G) \leq q$.

Here, the bounds are strict. For example, $b_n\gamma_G(C_6) = 0$ and

$b_n\gamma_G(K_p) = q$.

Result 3.6: $b_n\gamma_G(K_p - \{v\}) < b_n\gamma_G(K_p)$ for all $v \in V(K_p)$. This is true, since $K_p - \{v\} \cong K_{p-1}$.

Remark 3.7: The above result can be extended to any number of vertices. That is, $b_n\gamma_G(K_p - V') < b_n\gamma_G(K_p)$ for every proper subset V′ of $V(K_p)$ [since, $K_p - V' = K_{p-|V'|}$].

Theorem 3.8: If two adjacent vertices $u, v \in S$ for every γ_G-set S of G, then the edge $e = uv$ lies in every $b_n\gamma_G$-set of G.

Proof: Let $e = uv$ be an edge such that $u, v \in S$ for every γ_G-set S of G. Then, $\gamma_G[G - \{e\}] = \gamma_G(G)$ and so e lies in every $b_n\gamma_G$-set.

Remark 3.9: Converse of the above theorem is not true. Consider the graph G given in figure (2.1).

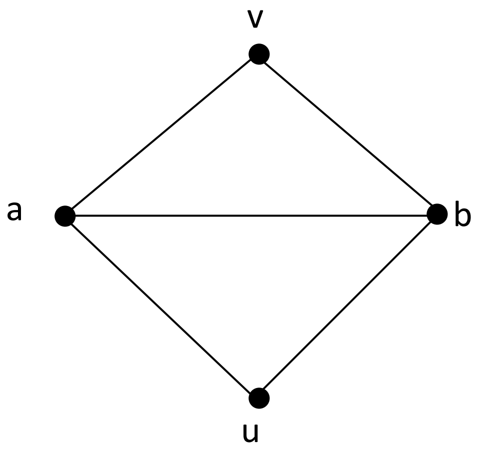

Figure (2.1)

Here, $S = \{u, v\}$ is the unique γ_G-set of G and so $\gamma_G(G) = 2$. Non-bondage (G, D)-sets ($b_n\gamma_G(G)$-sets) of G are $X_1 = \{ab, av\}$, $X_2 = \{ab, vb\}$, $X_3 = \{bu, ab\}$ and $X_4 = \{ab, au\}$. Here, the edge $e = ab$ lies in every nonbondage (G,D)-sets of G. But, the vertices of e(a and b) not belonging to S.

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