On Semigroup Ideals and Left Generalized (θ,θ)-4- Derivations in Prime Near-Rings

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Abstract

Let N be a near –ring and θ is a mapping on N .In this paper we introduce the concept of left generalized (θ,θ)--4-derivation in near-ring N and we show that prime near-ring N satisfying some identities involving left generalized (θ,θ)--4-derivation and semigroup ideals is a commutative ring .

Keywords

near-ring , prime near-ring , semigroup ideal , (θ, θ)-4-derivation , left generalized (θ, θ)-4-derivation .

I. INTRODUCTION

Let N be a near –ring and θ is a mapping on N . This paper consists of two sections . In section one , we recall some basic definitions and other concepts , which be used in our paper , we explain these concepts by examples and remarks . In section two , we introduce the notion of left generalized (θ, θ)-4-derivation in near-ring N and we determine some conditions of left generalized (θ, θ)-4-derivation and semigroup ideals which make prime near-ring commutative ring .

II. BASIC CONCEPTS

Definition 2.1:[1] A right near-ring (resp. a left near-ring) is a nonempty set N equipped with two binary operations + and . such that
(i)(N, +) is a group (not necessarily abelian)
(ii)(N, .) is a semigroup .
(iii) For all x,y,z $\in$N , we have
(x+y)z = xz + yz (resp. z(x+y) = zx + zy)

Example 2.2:[1] Let G be a group (not necessarily abelian) then all mapping of G into itself form a right near-ring M(G) with regard to point wise addition and multiplication by composite .

Lemma 2.3:[1] Let N be left (resp. right) near-ring , then
(i)x0 = 0 (resp. 0x = 0) for all x $\in$N .
(ii) – (xy) = x(-y) (resp. – (xy) = (-x)y) for all x,y $\in$N .

Definition 2.4:[1] A right near-ring (resp. left near-ring) is called zero symmetric right near-ring (resp. zero symmetric left near-ring) if x0 = 0 (resp. 0x = 0) , for all x $\in$N .

Definition 2.5:[2] Let {N_i } be a family of near-rings (i$\in$I , I is an indexing set) . N = N_1 x N_2 x…x N_n with regard to component wise addition and multiplication , N is called the direct product of near-rings N_i .

Definition 2.6:[2] A near-ring N is called a prime near-ring if aNb = {0}, where a,b $\in$N, implies that either a = 0 or b = 0 .

Definition 2.7:[2] Let N be a near-ring . The symbol Z will denote the multiplicative center of N , that is Z = {x$\in$N /xy = yx for all y $\in$N } .

Definition 2.8:[2] Let R be a ring . Define a Lie product [,] on R as follows
$[x,y] = xy - yx$, for all $x,y \in R$.

Properties 2.9:[2] Let R be a ring , then for all $x,y,z \in R$, we have :
1-$[x,yz] = y[x,z] + [x,y]z$
2-$[xy,z] = x[y,z] + [x,z]y$
3-$[x+y,z] = [x,z] + [y,z]$
4-$[x,y+z] = [x,y] + [x,z]$

Definition 2.10:[2] A nonempty subset U of N will be called a semigroup right ideal (resp. semigroup left ideal) if $UN \subset U$ (resp. $NU \subset U$) and if U is both semigroup right ideal and semigroup left ideal , it be called a semigroup ideal .

Remark 2.11:[3] Let N be a near-ring
(i)N x N x ….x N forms a near-ring with regard to component wise addition and component wise multiplication .
(ii)If U_1 , U_2 , ….,U_n be nonzero semigroup right ideals (resp. semigroup left ideals) of N , then $U_1 \times U_2 \times \ldots \times U_n$ forms a nonzero semigroup right ideals (resp. semigroup left ideas) of N x N x ….x N .

Definition 2.12:[3] Let R be a ring . Define a Jordan product on R as follows :
$a \circ b = ab + ba$, for all $a,b \in R$.

Properties 2.13 :[3] Let R be a ring , then for all $x,y,z \in R$, we have :

1- $x \circ (yz) = (x \circ y)z - y[x,z] = y(x \circ z) + [x,y]z$
2- $(xy) \circ z = x(y \circ z) - [x,z]y = (x \circ z)y + x[y,z]$

Lemma 2.14:[4] Let N be a prime near-ring. If $z \in Z \setminus \{0\}$ and x is an element of N such that $xz \in Z$ or $zx \in Z$, then $x \in Z$.

Lemma 2.15:[4] Let N be a prime near-ring and U be a nonzero semigroup ideal of N . If $x,y \in N$ and $xUy = \{0\}$, then $x = 0$ or $y = 0$.

Lemma 2.16:[4] Let N be a prime near-ring and Z contains a nonzero semigroup left ideal or nonzero semigroup right ideal , then N is a commutative ring .

Lemma 2.17:[4] Let d be n-derivation of a near-ring N , then
$d(Z,N,\ldots,N) \subseteq Z$.

Lemma 2.18:[4] Let N be a prime near-ring and d be a nonzero n-derivation of N . Let U_1 , U_2 , ….,U_n be a nonzero semigroup right ideals (resp. semigroup left ideals) of N , if $d(U_1 , U_2 , \ldots,U_n) \subseteq Z$ then N is a commutative ring .

Lemma 2.19:[4] Let N be a prime near-ring admitting a generalized n- derivation f with associated n-derivation d of N , then
$(d(x_1,x_2,\ldots,x_n) x_1' + x_1 f(x_1',x_2,\ldots,x_n)) y = d(x_1,x_2,\ldots,x_n) x_1'y + x_1 f(x_1',x_2,\ldots,x_n)y$,
$(d(x_1,x_2,\ldots,x_n) x_2' + x_2 f(x_1,x_2',\ldots,x_n))y = d(x_1,x_2,\ldots,x_n) x_2'y + x_2 f(x_1 ,x_2',\ldots,x_n)y$,
⋮
$(d(x_1,x_2,\ldots,x_n) x_n' + x_n f(x_1,x_2 ,\ldots,x_n'))y = d(x_1,x_2,\ldots,x_n) x_n'y + x_n f(x_1 ,x_2,\ldots,x_n')y$
hold for all $x_1 ,x_1',x_2 , x_2',\ldots, x_n , x_n' , y \in N$.

Definition 2.20:[4] Let N be a near-ring and n be a fixed positive integer . An n- additive mapping $d : N \times N \times \ldots \times N \rightarrow N$ is said to be n-derivation if the relations
$d(x_1x_1',x_2,\ldots,x_n) = d(x_1,x_2,\ldots,x_n) x_1' + x_1 d(x_1',x_2,\ldots,x_n)$
$d(x_1,x_2x_2',\ldots,x_n) = d(x_1,x_2,\ldots,x_n) x_2' + x_2 d(x_1,x_2',\ldots,x_n)$
⋮
$d(x_1,x_2,\ldots,x_nx_n') = d(x_1,x_2,\ldots,x_n) x_n' + x_n d(x_1,x_2,\ldots,x_n')$
hold for all $x_1 ,x_1',x_2 , x_2',\ldots,x_n, x_n' \in N$.

Definition 2.21:[4] Let N be a near-ring and d be n-derivation of N . An n-additive mapping f : N x N x….x N → N is said to be left generalized n- derivation d of N with associated n- derivation d if the relations :

$f(x_1x_1^{/},x_2,\ldots,x_n) = d(x_1,x_2,\ldots,x_n)\, x_1^{/} + x_1\, f(x_1^{/},x_2,\ldots,x_n)$

$f(x_1,x_2x_2^{/},\ldots,x_n) = d(x_1,x_2,\ldots,x_n)\, x_2^{/} + x_2\, f(x_1,x_2^{/},\ldots,x_n)$

⋮

$f(x_1,x_2,\ldots,x_nx_n^{/}) = d(x_1,x_2,\ldots,x_n)\, x_n^{/} + x_n\, f(x_1,x_2,\ldots,x_n^{/})$

hold for all $x_1, x_1^{/}, x_2, x_2^{/}, \ldots, x_n, x_n^{/} \in N$.

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III. MAIN RESULTS

First we introduce the basic definition in this paper

Definition 3.1: Let N be a near-ring and θ is a mapping on N . Let d: N x N x N x N → N be a (θ, θ)- 4-derivation of N . An 4-additive mapping f : N x N x N x N → N is said to be left generalized (θ, θ)- 4-derivation of N with associated (θ, θ)- 4- derivation d if the relations :

$f(x_1x_1^{/},x_2,x_3,x_4) = d(x_1,x_2,x_3,x_4)\,\theta(x_1^{/}) + \theta(x_1)\, f(x_1^{/},x_2,x_3,x_4)$

$f(x_1,x_2x_2^{/},x_3,x_4) = d(x_1,x_2,x_3,x_4)\,\theta(x_2^{/}) + \theta(x_2)\, f(x_1,x_2^{/},x_3,x_4)$

$f(x_1,x_2,x_3x_3^{/},x_4) = d(x_1,x_2,x_3,x_4)\,\theta(x_3^{/}) + \theta(x_3)\, f(x_1,x_2,x_3^{/},x_4)$

$f(x_1,x_2,x_3,x_4x_4^{/}) = d(x_1,x_2,x_3,x_4)\,\theta(x_4^{/}) + \theta(x_4)\, f(x_1,x_2,x_3,x_4^{/})$

hold for all $x_1, x_1^{/}, x_2, x_2^{/}, x_3, x_3^{/}, x_4, x_4^{/} \in N$.

we now explain this definition by the following example

Example 3.2 : Let S be a zero symmetric commutative near-ring .
Let us define

$$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} : x, y, z, 0 \in S \right\}.$$

It can be easily see that N is a non commutative zero symmetric left near-ring with respect to matrix addition and matrix multiplication .

Define d ,f : N x N x N x N → N such that

$$d\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & z_1 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & z_2 \end{pmatrix}, \begin{pmatrix} 0 & x_3 & y_3 \\ 0 & 0 & 0 \\ 0 & 0 & z_3 \end{pmatrix}, \begin{pmatrix} 0 & x_4 & y_4 \\ 0 & 0 & 0 \\ 0 & 0 & z_4 \end{pmatrix} \right) = \begin{pmatrix} 0 & x_1x_2x_3x_4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$f\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & z_1 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & z_2 \end{pmatrix}, \begin{pmatrix} 0 & x_3 & y_3 \\ 0 & 0 & 0 \\ 0 & 0 & z_3 \end{pmatrix}, \begin{pmatrix} 0 & x_4 & y_4 \\ 0 & 0 & 0 \\ 0 & 0 & z_4 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

And θ : N → N such that

$$\theta\left(\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} \right) = \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix}$$

It can be easily verified that d is (θ, θ)- 4-derivation of N and f is a left generalized (θ, θ)- 4- derivation of N with associated (θ, θ)- 4- derivation d .

Theorem 3.3 : Let N be a prime near-ring admitting a nonzero left generalized (θ, θ)- 4- derivation f with associated (θ, θ)- 4- derivation d of N , where θ is an automorphism on N . Let U_1 , U_2 , U_3 , U_4 be nonzero semigroup ideals of N . If $f(U_1, U_2, U_3, U_4) \subseteq Z$, then N is a commutative ring .

Proof : By our hypothesis , we get

$f(u_1u_1^{/}, u_2, u_3, u_4) = d(u_1,u_2,u_3,u_4)\,\theta(u_1^{/}) + \theta(u_1)\, f(u_1^{/},u_2,u_3,u_4) \in Z$

for all $u_1, u_1^{/} \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$. (3.1)

Now commuting equation (3.1) with the element $\theta(u_1)$ we have

$(d(u_1,u_2,u_3,u_4)\,\theta(u_1^{/}) + \theta(u_1)\, f(u_1^{/},u_2,u_3,u_4))\,\theta(u_1)$

$= \theta(u_1)\,(d(u_1,u_2,u_3,u_4)\,\theta(u_1^{/}) + \theta(u_1)\, f(u_1^{/},u_2,u_3,u_4))$

for all $u_1, u_1^{/} \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$.

Using Lemma 2.19 in previous equation , we get

$d(u_1,u_2,u_3,u_4)\,\theta(u_1^{/})\,\theta(u_1) + \theta(u_1)\, f(u_1^{/},u_2,u_3,u_4)\,\theta(u_1) =$

$\theta(u_1)\, d(u_1,u_2,u_3,u_4)\,\theta(u_1^{/}) + \theta(u_1)\,\theta(u_1)\, f(u_1^{/},u_2,u_3,u_4)$

for all $u_1, u_1^{/} \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$.

Using hypothesis in previous equation implies that

$d(u_1,u_2,u_3,u_4)\,\theta(u_1^{/})\,\theta(u_1) = \theta(u_1)\, d(u_1,u_2,u_3,u_4)\,\theta(u_1^{/})$

for all $u_1, u_1^{/} \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$.

Replacing $u_1^{/}$ by $u_1^{/}y$, where $y \in N$, in previous equation and using it again we get

$d(u_1,u_2,u_3,u_4)\,\theta(u_1^{/})\,(\theta(u_1y) - \theta(yu_1)) = 0$

for all $u_1, u_1^{/} \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$, $y \in N$.

i.e.; $d(u_1,u_2,u_3,u_4)\, U_1\,(\theta(u_1y) - \theta(yu_1)) = \{0\}$

for all $u_1 \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$, $y \in N$, using Lemma 2.15 , we conclude that for each $u_1 \in U_1$ either $u_1 \in Z$ or $d(u_1,u_2,u_3,u_4) = 0$ for all $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$. But using Lemma 2.17 lastly we get $d(u_1,u_2,u_3,u_4) \in Z$ for all $u_1 \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$,
i.e., $d(U_1, U_2, U_3, U_4) \subseteq Z$. If $d \neq 0$, then by using Lemma 2.18 we find that N is commutative ring . On the other hand if $d = 0$, then equation (3.1) takes the form $f(u_1u_1^{/}, u_2, u_3, u_4) = \theta(u_1)\, f(u_1^{/},u_2,u_3,u_4)$ for all $u_1, u_1^{/} \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$, by hypothesis and Lemma 2.14 , we get $u_1 \in Z$ for all $u_1 \in U_1$. Thus by Lemma 2.16 we conclude that N is a commutative ring .

Theorem 3.4 : Let N be a prime near-ring admitting a left generalized (θ, θ)- 4- derivation f with associated nonzero (θ, θ)- 4- derivation d of N , where θ is an automorphism on N . Let U_1 , U_2 , U_3 , U_4 be nonzero semigroup ideals of N . If $f([u_1, u_1^{/}], u_2, u_3, u_4) = 0$ for all $u_1, u_1^{/} \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$, then N is a commutative ring .

Proof : By our hypothesis , we have

$f([u_1, u_1^{/}], u_2, u_3, u_4) = 0$ for all $u_1, u_1^{/} \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$. (3.2)

Replacing $u_1^{/}$ by $u_1 u_1^{/}$ in (3.2) and using it again we get

$d(u_1,u_2,u_3,u_4)\,[\theta(u_1), \theta(u_1^{/})] = 0$ for all $u_1, u_1^{/} \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$.

Therefore

$d(u_1,u_2,u_3,u_4)\,\theta(u_1)\,\theta(u_1^{/}) = d(u_1,u_2,u_3,u_4)\,\theta(u_1^{/})\,\theta(u_1)$

for all $u_1, u_1^{/} \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$. (3.3)

Replacing $u_1^{/}$ by $u_1^{/}r$, where $r \in N$, in equation (3.3) and using it again we get

$d(u_1,u_2,u_3,u_4)\, U_1\,[\theta(u_1), \theta(r)] = \{0\}$ for all $u_1 \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$, $r \in N$. By using Lemma 2.15 , we conclude that for each $u_1 \in U_1$ either $u_1 \in Z$ or $d(u_1,u_2,u_3,u_4) = 0$ for all $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$. But using Lemma 2.17 lastly we get $d(u_1,u_2,u_3,u_4) \in Z$ for all $u_1 \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$, so we get

$d(U_1, U_2, U_3, U_4) \subseteq Z$. By Lemma 2.18 , we find that N is a commutative ring .

Theorem 3.5 : Let N be a prime near-ring admitting a left generalized (θ, θ)- 4- derivation f associated with nonzero (θ, θ)- 4- derivation d of N , where θ is an automorphism on N . Let U_1 , U_2 , U_3 , U_4 be nonzero semigroup ideals of N . If $f([u_1, u_1^{/}], u_2, u_3, u_4) = \pm [\theta(u_1), \theta(u_1^{/})]$ for all $u_1, u_1^{/} \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$, then N is a commutative ring .

Proof : By our hypothesis , we have

$f([u_1, u_1^{/}], u_2, u_3, u_4) = \pm [\theta(u_1), \theta(u_1^{/})]$ for all $u_1, u_1^{/} \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$.

Replacing $u_1^{/}$ by $u_1 u_1^{/}$, in preceding equation and using it again we get

$d(u_1,u_2,u_3,u_4)[\theta(u_1), \theta(u_1^{/})] = 0$ for all $u_1, u_1^{/} \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$.

Therefore

$d(u_1,u_2,u_3,u_4)\,\theta(u_1)\,\theta(u_1^{/}) = d(u_1,u_2,u_3,u_4)\,\theta(u_1^{/})\,\theta(u_1)$ for all $u_1, u_1^{/} \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$. (3.4)

Which is identical with the equation (3.3) in Theorem 3.4 . Now arguing in the same way in the Theorem 3.4 we conclude that N is a commutative ring .

Theorem 3.6 : Let N be a prime near-ring admitting a left generalized (θ, θ)- 4- derivation f associated with nonzero (θ, θ)- 4- derivation d of N , where θ is an automorphism on N . Let U_1, U_2, U_3, U_4 be nonzero semigroup ideals of N . If $f([u_1, u_1^{/}], u_2, u_3, u_4) = \pm (\theta(u_1) \circ \theta(u_1^{/}))$ for all $u_1, u_1^{/} \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$, then N is a commutative ring .

Proof : By our hypothesis , we have ,

$f([u_1, u_1^{/}], u_2, u_3, u_4) = \pm (\theta(u_1) \circ \theta(u_1^{/}))$

for all $u_1, u_1^{/} \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$. (3.5)

Substituting $u_1 u_1^{/}$ for $u_1^{/}$ in (3.5) , we obtain

$f(u_1[u_1, u_1^{/}], u_2, u_3, u_4) = \pm \theta(u_1)(\theta(u_1) \circ \theta(u_1^{/}))$ for all $u_1, u_1^{/} \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$.

Therefore

$d(u_1,u_2,u_3,u_4)[\theta(u_1), \theta(u_1^{/})] + \theta(u_1) f([u_1, u_1^{/}], u_2, u_3, u_4)$

$= \pm\theta(u_1)(\theta(u_1) \circ \theta(u_1^{/}))$ for all $u_1, u_1^{/} \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$.

Using (3.5) in previous equation , we get

$d(u_1,u_2,u_3,u_4)\,\theta(u_1)\,\theta(u_1^{/}) = d(u_1,u_2,u_3,u_4)\,\theta(u_1^{/})\,\theta(u_1)$ for all $u_1, u_1^{/} \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$.

Which is identical with the equation (3.3) in Theorem 3.4 . Now arguing in the same way in the Theorem 3.4 we conclude that N is a commutative ring .

IV. Conclusions

In present paper we introduce the notion of left generalized (θ, θ)- 4- derivation in near-ring and we see that a near-ring can be make commutative with the help of left generalized (θ, θ)- 4- derivation and other conditions .

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