Pseudo Arithmetic Operations on Fuzzy Measure

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Abstract:

In this paper the axiomatic approach to the Pseudo arithmetical operations of pseudo addition and pseudo multiplication is discussed.Also several properties of $\oplus$ *and* $\otimes$ *their consequences are discussed and illustrated by examples.*

Keywords:

Pseudo addition,pseudo multiplication,semi ring,distributivity , decomposable measure , Lebesgue measure.

Introduction:

Taking into account human subjective measures in engineering,science fuzzy measures have been intensively discussed. Since sugeno (5) defined a fuzzy measure as a measure having monotonicity property instead of additivity. Weber(6) proposed $\perp$ decomposable measures where the additivity of measures is weakend t-conorm $\perp$ is an appropriate semigroup operation in [0,1] and $\perp$ is considered to be a generalization of addition $\perp$ decomposable measures can be written as $m(A \cup B) = m(A) \perp m(B)$

For the archimedian case $\perp$ is written as

$a \perp b = g^{(-1)} (g(a) + g(b))$

where $g^{(-1)}$ is a pseudo inverse of g.

if a law of composition

$\oplus : X \times X \rightarrow X$ is defined on a set X and its fulfills the laws of assciativity then $(X,\oplus)$ is a semi group.

Lebesgue measure is defined on a set $[0, \infty]$. $([0, \infty],+)$ is a semi group and the operation + has no inverse element . In sec 2 define an integral by pseudo addition

$\oplus$ and pseudo multiplication $\otimes$ that are distributive and associative semiring operations .In sec 3 define fuzzy measure derived from pseudo addition.

1.PREMILINARIES:

1.1Definition:

Let [a,b] be a closed real interval and `$\oplus : [a,b] \times [a,b] \rightarrow [a,b]$

be a 2 place function satisfying the following conditions

(1) $\oplus$ is commutative
(2) $\oplus$ is non decreasing in each place
(3) $\oplus$ is associative
(4) $\oplus$ has either a (or) b as zero element

ie) either ⊕ (a, x) = x (or)

⊕ (b, x) = x

⊕ will be called a pseudo addition.

1.2 Definition:

A pseudo multiplication ⊗ is a 2 place function

⊗: [a,b] × [a,b]→[a,b]

Satisfying the following conditions

(1) ⊗ is commutative
(2) ⊗ is non decreasing in each place
(3) ⊗ is associative
(4) There exist a unit element e ∈ [a,b]
⊗(x,e) = x ∀ x∈ [a,b]

1.3 theorem:

If the function ⊕ is continuous and strictly increasing in

(a,b) × (a,b) then there exists a monotone function

g : [a,b] →[0 ,∞] such that either g(a) =0 (or) g (b) =0

$\oplus(x,y) = g^{-1}(g(x) + g(y))$

Proof:

Trivial from aczel's theorem (5)

1.4 REMARK :

By applying g to both sides

We have

g(⊕ (x ,y)) = g(x) +g (y) ∀ x ,y in [a, b]. hence g is nothing but an isomorphism of semigroup [a,b] relative to ⊕ with a semigroup of the non negative real numbers relative to addition. g is called a generator of ⊕.

2.PSEUDO ADDITIONS AND PSEUDO MULTIPLICATIONS:

2.1 Definition:

A pseudo multiplication ⊗ with the generator g of strict pseudo addition ⊕ is defined as ⊗(x ,y) = g^{-1} (g(x) .g(y))

g (x).g(y) is always in [0,∞] because g is a monotone function from [a ,b] to

[0,∞] .Thus g^{-1} is always exists and g is an isomorphism of semigroup ([a,b] ,⊗) with a subgroup ([0,∞] ,x).

2.2 REMARK:

For the sake of simplicity we write ⊕ (x, y) as x ⊕ y and ⊗ (x,y) as x ⊗ y

Respectively. Distributive pseudo multiplication $\otimes$ has the distributive property with pseudo addition $\oplus$

$x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$

$(y \oplus z) \otimes x = (y \otimes x) \oplus (z \otimes x)$

2.3 EXAMPLE:

$x \vee y = \max(x,y)$

$x \wedge y = \min(x,y)$

are continuous nonstrict pseudo addition $\oplus$ and pseudo multiplication $\otimes$ in

$[a, b] \subset [-\infty, +\infty]$ respectively.

2.4 THEOREM:

When any pseudo addition, $\oplus$,(pseudo multiplication $\oplus$) defined on

$[a_1 \times b_1] \times [a_1 \times b_1]$ a strict monotone function g_1 from $[a_2, b_2]$ onto $[a_1, b_1]$

are given a 2 place function $\oplus_2$

$\oplus_2 (x_2, y_2) = g_1^{-1} (\oplus_1 (g_1(x_2), g_1(y_2)))$

$\otimes_2 (x_2, y_2) = g_1^{-1} (\otimes_1 (g_1(x_2), g_1(y_2)))$

is a pseudo addition.

proof:

The commutative and associative property of $\oplus_2$ is a immediate consequence of

That of $\oplus_1$ similarly the non decreasingness of $\oplus_2$ follows directly from the non decreasingness of $\oplus_1$ and the monotonicity of g_1 as for the zero element .if a_1 is a zero element and g_1 is an increasing function

Then $g_1(a_2) = a_1$

For all $x \in [a_2, b_2]$

$\oplus_2 (x, a_2) = g_1^{-1} (\oplus_1 (g_1(x), a_1))$

$= g_1^{-1} (g_1(x))$

$= x$

2.5 **Definition:**

The pseudo operation on the set I is a binary operation $* : I \times I \rightarrow I$

Which is commutative associative “positively non-decreasing” in the sense that

For all $u \in I^*$,$x \leq y$ implies $x * u \leq y * u$ and for which there exists a neutral element $e \in I$

2.6 Definition:

The element u∈I is a null element of the operation $* : I^2 \to I$ if for any $x \in I$

$x*u = u*x = u$ holds.

3.SOME ADDITIONAL PROPERTIES OF PSEUDO MEASURE:

3.1THEOREM:

Let ⊗ be a continuous pseudo multiplication for the sub measure 'm' and a family of functions $f_j : \Omega \to [0,\infty)$, j∈J

It holds

$$\int_{\Omega}^{sup} (\sup_{j \in J} f_j) \otimes dm = \sup_{j \in J} \int_{\Omega}^{sup} (f_j \otimes dm)$$

Let us consider now the semi ring ([0,∞], min,⊗)

With continuous operation ⊗. Let $\boldsymbol{\phi}$ be a density function of inf measure m.

3.2 THEOREM:

For the inf decomposable measure m and a family of functions $f_j : \Omega \to (0,\infty]$

$j \in J$ it holds

$$\int_{\Omega}^{sup} (\sup_{j \in J} f_j) \otimes dm = \sup_{j \in J} \int_{\Omega}^{sup} (f_j \otimes dm)$$

Proof:

$$\int_{\Omega}^{inf} (\inf_{j \in J} f_j) \otimes dm = \inf_{w \in \Omega} \left\{ \left(\inf_{j \in J} f_j \right) w \otimes \boldsymbol{\phi}(w) \right\}$$

$$= \inf_{w \in \Omega} \left\{ \inf_{j \in J} f_j (w) \otimes \boldsymbol{\phi}(w) \right\}$$

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$$= \inf_{j \in J} \int_{\Omega}^{inf} f_j \otimes dm$$

3.3PROPOSITION:

If the pseudo multiplication ⊗ has the property (*) then for $f : \Omega \to [0,\infty]$

And $h : \Omega \to [0,\infty]$ holds

$$\int_{\Omega}^{sup} (f + h) \otimes dm \le \int_{\Omega}^{sup} f \otimes dm + \int_{\Omega}^{sup} h \otimes dm$$

Proof:

$\int_{\Omega}^{sup} (f + h) \otimes dm$ = sup $_{w\in\Omega}$ {(f + h) w ⊗ (w)}

= sup $_{w\in\Omega}$ {(f(w) + h(w) ⊗ $\boldsymbol{\phi}$(w)}

≤ sup$_{w\in\Omega}$ {f(w) ⊗ $\boldsymbol{\phi}$(w) + h(w) ⊗ $\boldsymbol{\phi}$(w)}

≤ sup$_{w\in\Omega}$ {f(w) ⊗ $\boldsymbol{\phi}$(w) }+ sup$_{w\in\Omega}$ {h(w) ⊗ $\boldsymbol{\phi}$(w)}

= $\int_{\Omega}^{sup} f \otimes dm + \int_{\Omega}^{sup} h \otimes dm$

3.4 PROPOSITION:

If the pseudo multiplication has the property f : Ω→ [0,∞] and

h : Ω→[0,∞] holds

$$\left| \int_{\Omega}^{sup} f \otimes dm + \int_{\Omega}^{sup} h \otimes dm \right| \leq \int_{[0,\infty]}^{sup} \left| f - h \right| \otimes dm$$

Proof:

Suppose that

$$\int_{\Omega}^{sup} f \otimes dm \geq \int_{\Omega}^{sup} h \otimes dm$$

$$\left| \int_{[0,\infty]}^{sup} f \otimes dm - \int_{[0,\infty]}^{sup} h \otimes dm \right| = \int_{[0,\infty]}^{sup} f \otimes dm - \int_{[0,\infty]}^{sup} h \otimes dm$$

positive function f and h satisfies f(w) = | f-h| (w) +h(w).density function $\boldsymbol{\phi}$ is also

positive.so that f(w) ⊗ $\boldsymbol{\phi}$(w) ≤ (|f-h| (w) + h(w)) ⊗ $\boldsymbol{\phi}$(w)

hence using above proposition

$\int_{\Omega}^{sup} f \otimes dm$ = sup $_{w\in\Omega}$ { f(w) ⊗ $\boldsymbol{\phi}$(w) }

≤ sup $_{w\in\Omega}$ { | f-h| (w) + h(w) ⊗ $\boldsymbol{\phi}$(w) }

$= \int_{\Omega}^{\sup}(|f - h| + h) \otimes dm$

$\leq \int_{\Omega}^{Sup} |f - h| \otimes dm + \int_{\Omega}^{sup} h \otimes dm$

So that

$$\left| \int_{[0,\infty]}^{sup} f \otimes dm - \int_{[0,\infty]}^{sup} h \otimes dm \right| = \int_{[0,\infty]}^{sup} f \otimes dm - \int_{[0,\infty]}^{sup} h \otimes dm$$

$$\leq \int_{[0,\infty]}^{\sup}(|f - h| \otimes dm$$

3.5 Definition:

A fuzzy integral based on ⊕ decmposable measure m is defined as

$S = \sum_{i=1}^{n} a_{I}.1_{Bi}$

With disjoint B_i

$\int_B S \otimes m = \oplus_{i=1}^{n} a_i \otimes m(B_i)$, $a \leq a_i \leq b$

3.6 THEOREM:

The functional $F_B : u \rightarrow \int_B u \otimes m$

has the following properties

(i) $F_B(u \oplus v) = F_B(u) \oplus F_B(v)$
(ii) $F_B(s \otimes u) = s \otimes F_B(u)$
(iii) $B_i \cap B_j = \phi \Rightarrow F_{Bi \cup Bj}(u) = F_{Bi}(u) \oplus F_{Bj}(u)$

Proof:

(i) $F_B(u \oplus v) = g^{-1}(\int_B (g(u) + g(v))\, d(g \circ m)$

$= g^{-1}(\int_B (g(u)\, d(g \circ m) + \int_B (g(v)\, d(g \circ m)$

$= g^{-1}(g(F_B(u)) + g(F_B(v)))$

$= F_B(u) + F_B(v)$

(ii) $F_B(s \otimes u) = g^{-1}(\int_B g(s).g(u)d(g \circ m))$

$= g^{-1}(g(s) \int_B g(u)d(g \circ m))$

$= g^{-1}(g(s) \circ g(F_B(u)))$

$= s \otimes F_B(u)$

(iii) $F_{Bi \cup Bj}(u) = g^{-1}(\int_{Bi \cup Bj} g(u)d(g \circ m))$

$= g^{-1}(\int_{Bi} g(u)d(g \circ m) + \int_{Bj} g(u)d(g \circ m))$

$= g^{-1}(g(F_{Bi}(u)) + g(F_{Bj}(u)))$

$= F_{Bi}(u) \oplus F_{Bj}(u)$

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