On Certain Expansion for Generalized Multivariable Gimel-Function

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ABSTRACT
The aim of the present paper is to evaluate four finite integrals involving the product of Jacobi polynomials and the generalized multivariable Gimel-function with general arguments. These integrals have been utilized to derive the expansion formula for generalized multivariable Gimel-function in series involving Jacobi polynomials.

KEYWORDS : Multivariable Gimel-function, multiple integral contours, Jacobi polynomials, expansion serie, finite integrals.

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1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C},\mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$.

In this paper, we establish four single Fourier-Jacobi expansions formulae for generalized multivariable Gimel-function.

We define a generalized transcendental function of several complex variables.

$$\gimel(z_1,\cdots,z_r) = \gimel^{m_2,n_2;m_3,n_3;\cdots;m_r,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array}\right|$$

$$[(a_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},$$
$$[(b_{2j};\beta^{(1)}_{2j},\beta^{(2)}_{2j};B_{2j})]_{1,m_2},[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{m_2+1,q_{i_2}},[(b_{3j};\beta^{(1)}_{3j},\beta^{(2)}_{3j},\beta^{(3)}_{3j};B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(a_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{m_3+1,q_{i_3}};\cdots;[(b_{rj};\beta^{(1)}_{rj},\cdots,\beta^{(r)}_{rj};B_{rj})_{1,m_r}],$$

$$[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{n_r+1,p_r}]:\ [(c^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]$$
$$[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{m_r+1,q_r}]:[(d^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]$$

$$\left.\begin{array}{l};\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})_{m^{(r)}+1,p^{(r)}_i}] \\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})_{n^{(r)}+1,q^{(r)}_i}]\end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{m_2}\Gamma^{B_{2j}}(b_{2j}-\sum_{k=1}^{2}\beta^{(k)}_{2j}s_k)\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha^{(k)}_{2j}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha^{(k)}_{2ji_2}s_k)\prod_{j=m_2+1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta^{(k)}_{2ji2}s_k)]}$$

$$\frac{\prod_{j=1}^{m_3}\Gamma^{B_{3j}}(b_{3j}-\sum_{k=1}^{3}\beta_{3j}^{(k)}s_k)\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=m_3+1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{m_r}\Gamma^{B_{rj}}(b_{rj}-\sum_{k=1}^{r}\beta_{rj}^{(k)}s_k)\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=m_r+1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)})]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $m_2,n_2,\cdots,m_r,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2\leqslant q_{i_2},0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant m_r\leqslant q_{i_r},0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$

$0\leqslant n^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant p_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kj}^{(l)},B_{kj}\in\mathbb{R}^+;(j=1,\cdots,m_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)} \in \mathbb{C}; (j=1,\cdots,n_k);(k=1,\cdots,r); d_j^{(k)} \in \mathbb{C}; (j=1,\cdots,m_k);(k=1,\cdots,r)$.

$a_{kji_k} \in \mathbb{C}; (j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r)$.
$b_{kji_k} \in \mathbb{C}; (j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r)$.

$d_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

The contour L_k is in the $s_k(k=1,\cdots,r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k\right)(j=1,\cdots,n_2), \Gamma^{A_{3j}}\left(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k\right)$ $(j=1,\cdots,n_3),\cdots,\Gamma^{A_{rj}}\left(1-a_{rj}+\sum_{i=1}^{r}\alpha_{rj}^{(i)}\right)(j=1,\cdots,n_r), \Gamma^{C_j^{(k)}}\left(1-c_j^{(k)}+\gamma_j^{(k)}s_k\right)(j=1,\cdots,n^{(k)})(k=1,\cdots,r)$to the right of the contour L_k and the poles of $\Gamma^{B_{2j}}\left(b_{2j}-\sum_{k=1}^{2}\beta_{2j}^{(k)}s_k\right)(j=1,\cdots,m_2), \Gamma^{B_{3j}}\left(b_{3j}-\sum_{k=1}^{3}\beta_{3j}^{(k)}s_k\right)(j=1,\cdots,m_3)$ $,\cdots,\Gamma^{B_{rj}}\left(b_{rj}-\sum_{i=1}^{r}\beta_{rj}^{(i)}\right)(j=1,\cdots,m_r), \Gamma^{D_j^{(k)}}\left(d_j^{(k)}-\delta_j^{(k)}s_k\right)(j=1,\cdots,m^{(k)})(k=1,\cdots,r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2}A_i^{(k)}\pi$ where

$$A_i^{(k)} = \sum_{j=1}^{m^{(k)}} D_j^{(k)}\delta_j^{(k)} + \sum_{j=1}^{n^{(k)}} C_j^{(k)}\gamma_j^{(k)} - \tau_{i^{(k)}}\left(\sum_{j=m^{(k)}+1}^{q_i^{(k)}} D_{ji^{(k)}}^{(k)}\delta_{ji^{(k)}}^{(k)} + \sum_{j=n^{(k)}+1}^{p_i^{(k)}} C_{ji^{(k)}}^{(k)}\gamma_{ji^{(k)}}^{(k)}\right)+$$

$$\sum_{j=1}^{n_2} A_{2j}\alpha_{2j}^{(k)} + \sum_{j=1}^{m_2} B_{2j}\beta_{2j}^{(k)} - \tau_{i_2}\left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2}\alpha_{2ji_2}^{(k)} + \sum_{j=m_2+1}^{q_{i_2}} B_{2ji_2}\beta_{2ji_2}^{(k)}\right)+\cdots+$$

$$\sum_{j=1}^{n_r} A_{rj}\alpha_{rj}^{(k)} + \sum_{j=1}^{m_r} B_{rj}\beta_{rj}^{(k)} - \tau_{i_r}\left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r}\alpha_{rji_r}^{(k)} + \sum_{j=m_r+1}^{q_{i_r}} B_{rji_r}\beta_{rji_r}^{(k)}\right) \tag{1.4}$$

Following the lines of Braaksma ([3] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,), max(\,|z_1|,\cdots,|z_r|\,) \rightarrow 0$

$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,), min(\,|z_1|,\cdots,|z_r|\,) \rightarrow \infty$ where $i=1,\cdots,r$:

$\alpha_i = \min_{\substack{1\leqslant j\leqslant m_i \\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}} + D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)$ and $\beta_i = \max_{\substack{1\leqslant j\leqslant n_i \\ 1\leqslant j\leqslant n^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} A_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}} + C_j^{(i)}\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)$

Remark 1.
If $m_2=n_2=\cdots=m_{r-1}=n_{r-1}=p_{i_2}=q_{i_2}=\cdots=p_{i_{r-1}}=q_{i_{r-1}}=0$ and $A_{2j}=B_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=B_{rj}=A_{rji_r}=B_{rji_r}=1$, then the generalized multivariable Gimel-function reduces in the generalized multivariable Aleph- function (extension of multivariable Aleph-function defined by Ayant [2]).

Remark 2.
If $m_2=n_2=\cdots=m_r=n_r=p_{i_2}=q_{i_2}=\cdots=p_{i_r}=q_{i_r}=0$ and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2= =\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then the generalized multivariable Gimel-function reduces in a generalized multivariable I-function (extension of multivariable I-function defined by Prathima et al. [5]).

Remark 3.
If $A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in generalized of multivariable I-function (extension of multivariable I-function defined by Prasad [4]).

Remark 4.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the generalized multivariable H-function (extension of multivariable H-function defined by Srivastava and panda [7,8]).

In your investigation, we shall use the following notations.

$$\mathbb{A} = [(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.5}$$

$$\mathbf{A} = [(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A = [(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \tag{1.7}$$

$$\mathbb{B} = [(b_{2j};\beta_{2j}^{(1)},\beta_{2j}^{(2)};B_{2j})]_{1,m_2},[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{m_2+1,q_{i_2}},[(b_{3j};\beta_{3j}^{(1)},\beta_{3j}^{(2)},\beta_{3j}^{(3)};B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{m_3+1,q_{i_3}};\cdots;[(\mathrm{b}_{(r-1)j};\beta_{(r-1)j}^{(1)},\cdots,\beta_{(r-1)j}^{(r-1)};B_{(r-1)j})_{1,m_{r-1}}],$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})_{m_{r-1}+1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B} = [(\mathrm{b}_{rj};\beta_{rj}^{(1)},\cdots,\beta_{rj}^{(r)};B_{rj})_{1,m_r}],[\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{m_r+1,q_{i_r}}] \tag{1.9}$$

$$\mathrm{B} = [(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}];\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \tag{1.10}$$

$$U = m_2,n_2;m_3,n_3;\cdots;m_{r-1},n_{r-1};V = m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X = p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y = p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

2. Required results.

In this section, we give four finite integrals involving Jacobi polynomials, see Anandani and Shrivastava [1].

Lemma1.

$$\int_{-1}^{1} x^{\lambda}(1-x)^{\sigma}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)\mathrm{d}x = (-)^n 2^{\alpha+\sigma+1}$$

$$\frac{\Gamma(\alpha+n+1)\Gamma(\alpha-\beta+1)\Gamma(\sigma+1)}{n!\Gamma(\sigma-\beta-n+1)\Gamma(\sigma+\alpha+n+2)}{}_3F_2\left[\begin{array}{c|c} \text{-}\lambda,\sigma-\beta+1,\sigma+1 & \\ . & 2 \\ \sigma-\alpha-n+1,\sigma+\alpha+n+2 & \end{array}\right] \tag{2.1}$$

Lemma2.

$$\int_{-1}^{1} x^{\lambda}(1-x)^{\rho}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)\mathrm{d}x = (-)^n 2^{\rho+\beta+1}$$

$$\frac{\Gamma(\beta+n+1)\Gamma(\rho-\beta+1)\Gamma(\rho-\beta+1)\Gamma(\rho+1)}{n!\Gamma(\rho-\alpha-n+1)\Gamma(\rho+\beta+n+2)}{}_3F_2\left[\begin{array}{c|c} \text{-}\lambda,\rho-\beta+1,\rho+1 & \\ . & 2 \\ \rho-\alpha-n+1,\rho+\beta+n+2 & \end{array}\right] \tag{2.2}$$

3. Main integrals.

In this section, we prove the following four finite integrals.

Theorem 1.

$$\int_{-1}^{1} x^{\lambda}(1-x)^{\alpha}(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)\gimel\left(z_1\frac{(1-x)^{u_1}}{x^{v_1}},\cdots,z_r\frac{(1-x)^{u_r}}{x^{v_r}}\right)\mathrm{d}x =$$

$$\frac{(-)^n 2^{\sigma+\alpha+1}\Gamma(1+\alpha+n)}{n!}\sum_{k=0}^{\infty}\frac{2^k}{k!}\gimel^{U;m_r+2,n_r+3:V}_{X;p_{i_r}+5,q_{i_r}+5,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} 2^{u_1}z_1 & \mathbb{A};A_1,\mathbf{A},A_2:A \\ . & . \\ . & . \\ 2^{u_r}z_r & \mathbb{B};B_1,\mathbf{B},B_2:B \end{array}\right) \tag{3.1}$$

where

$$A_1 = (1-k;v_1,\cdots,v_r;1),(\beta-\sigma-k;u_1,\cdots,u_r;1),(-\sigma-k;u_1,\cdots,u_r;1) \tag{3.2}$$

$$A_2 = (1;v_1,\cdots,v_r;1),(1+\lambda-k;v_1,\cdots,v_r;1) \tag{3.3}$$

$$B_1 = (1-k;v_1,\cdots,v_r;1),(1+\lambda;v_1,\cdots,v_r;1) \tag{3.4}$$

$$B_2 = (1;v_1,\cdots,v_r;1),(n+\beta-\sigma-k;u_1,\cdots,u_r;1),(-1-\sigma-k-\alpha-n;u_1,\cdots,u_r;1) \tag{3.5}$$

provided

$$Re(\alpha+\beta),u_i,v_i>0(i=1,\cdots,r),\ Re(1+\lambda)-\sum_{i=1}^{r}v_i\min_{\substack{1\leqslant j\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0.$$

$$Re(1-k)-\sum_{i=1}^{r}v_i\min_{\substack{1\leqslant j\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0.$$

$$Re(\sigma+k-\alpha)+\sum_{i=1}^{r}u_i\min_{\substack{1\leqslant j\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0 \text{ and}$$

$\left|arg\left(z_i\frac{(1-x)^{u_i}}{x^{v_i}}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Proof
To prove (2.1), expressing the generalized multivariable Gimel-function with the help of (1.1), interchanging the order of integration which is justified under the conditions mentioned above, evaluating the inner integral with the help of the

lemma 1, and interpreting the Mellin-Barnes multiple integrals contour in terms of the multivariable Gimel-function, we get the desired result (3.1).

Theorem 2.

$$\int_{-1}^{1} x^{\lambda}(1-x)^{\alpha}(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)\gimel\left(z_1x^{u_1}(1+x)^{v_1},\cdots,z_rx^{u_r}(1+x)^{v_r}\right)\mathrm{d}x=$$

$$\frac{(-)^n2^{\sigma+\alpha+1}\Gamma(1+\alpha+n)}{n!}\sum_{k=0}^{\infty}\frac{2^k}{k!}\gimel^{U;m_r+1,n_r+2:V}_{X;p_{i_r}+3,q_{i_r}+3,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} 2^{v_1}z_1 & \mathbb{A};(\beta-\sigma-k;v_1,\cdots,v_r;1),(k-\sigma;v_1,\cdots,v_r;1),\mathbf{A},(-\lambda;u_1,\cdots,u_r;1):A \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{v_r}z_r & \mathbb{B};(-\lambda+k;u_1,\cdots,u_r;1),\mathbf{B},(\beta+n-\sigma-k;v_1,\cdots,v_r;1),(-1-n-\alpha-\sigma-k;v_1,\cdots,v_r;1):B \end{array}\right) \tag{3.6}$$

provided

$$Re(\alpha+1),u_i,v_i>0(i=1,\cdots,r),\ Re(\lambda+1)+\sum_{i=1}^{r}u_i\min_{\substack{1\leqslant j\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0.$$

$$Re(1+\sigma)+\sum_{i=1}^{r}v_i\min_{\substack{1\leqslant j\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0$$ and

$|arg\left(z_i(1+x)^{v_i}x^{u_i}\right)|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Theorem 3.

$$\int_{-1}^{1} x^{\lambda}(1-x)^{\rho}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)\gimel\left(z_1x^{u_1}(1-x)^{v_1},\cdots,z_rx^{u_r}(1-x)^{v_r}\right)\mathrm{d}x=$$

$$\frac{(-)^n2^{\rho+\beta+1}\Gamma(1+\beta+n)}{n!}\sum_{k=0}^{\infty}\frac{2^k}{k!}\gimel^{U;m_r+1,n_r+2:V}_{X;p_{i_r}+3,q_{i_r}+3,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} 2^{v_1}z_1 & \mathbb{A};(\beta-\rho-k;v_1,\cdots,v_r;1),(-k-\rho;v_1,\cdots,v_r;1),\mathbf{A},(-\lambda;u_1,\cdots,u_r;1):A \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{v_r}z_r & \mathbb{B};(-\lambda+k;u_1,\cdots,u_r;1),\mathbf{B},(\alpha+n-\rho-k;v_1,\cdots,v_r;1),(-1-n-\beta-\rho-k;v_1,\cdots,v_r;1):B \end{array}\right) \tag{3.7}$$

provided

$$Re(\beta+1),u_i,v_i>0(i=1,\cdots,r),\ Re(1+\lambda)+\sum_{i=1}^{r}u_i\min_{\substack{1\leqslant j\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0.$$

$$Re(1+\rho)+\sum_{i=1}^{r}v_i\min_{\substack{1\leqslant j\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0$$ and

$|arg\left(z_i(1-x)^{v_i}x^{u_i}\right)|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Theorem 4.

$$\int_{-1}^{1} x^{\lambda}(1-x)^{\rho}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)\gimel\left(z_1\frac{(1-x)^{u_1}}{x^{v_1}},\cdots,z_r\frac{(1-x)^{u_r}}{x^{v_r}}\right)\mathrm{d}x =$$

$$\frac{(-)^n 2^{\rho+\beta+1}\Gamma(1+\beta+n)}{n!}\sum_{k=0}^{\infty}\frac{2^k}{k!}\gimel_{X;p_{i_r}+5,q_{i_r}+5,\tau_{i_r}:R_r:Y}^{U;m_r+2,n_r+3:V}\left(\begin{array}{c|c} 2^{u_1}z_1 & \mathbb{A};A'_1,\mathbf{A},A_2:A \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{u_r}z_r & \mathbb{B};B_1,\mathbf{B},B'_2:B \end{array}\right) \tag{3.8}$$

where

$$A'_1=(1-k;v_1,\cdots,v_r;1),(\beta-\rho-k;u_1,\cdots,u_r;1),(1+\lambda-k;u_1,\cdots,u_r;1) \tag{3.9}$$

$$B'_2=(1;v_1,\cdots,v_r;1),(n+\alpha-\rho-k;u_1,\cdots,u_r;1),(-1-\rho-k-\beta-n;u_1,\cdots,u_r;1) \tag{3.10}$$

provided

$$Re(1+\alpha),u_i,v_i>0(i=1,\cdots,r),\ Re(1-k)-\sum_{i=1}^{r}v_i\min_{\substack{1\leqslant j\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0.$$

$$Re(1+\lambda)-\sum_{i=1}^{r}v_i\min_{\substack{1\leqslant j\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0.$$

$$Re(1+\sigma-\beta+k)+\sum_{i=1}^{r}u_i\min_{\substack{1\leqslant j\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0$$ and

$\left|arg\left(z_i\frac{(1-x)^{u_i}}{x^{v_i}}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

To prove the theorem3 2 to 4, we use the similar method that theorem 1.

4. Expansion formulae.

In this section, we establish the following expansions for the generalized multivariable Gimel-function in series involving product of Jacobi polynomials and generalized multivariable Gimel-function.

Theorem 5.

$$x^{\lambda}(1-x)^{\rho}\gimel\left(z_1\frac{(1-x)^{u_1}}{x^{v_1}},\cdots,z_r\frac{(1-x)^{u_r}}{x^{v_r}}\right)=\sum_{n,k=0}^{\infty}\frac{(-)^n2^{k+\rho}(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)}{k!\Gamma(1+\alpha+n)}P_n^{(\alpha,\beta)}(x)$$

$$\gimel_{X;p_{i_r}+5,q_{i_r}+5,\tau_{i_r}:R_r:Y}^{U;m_r+2,n_r+3:V}\left(\begin{array}{c|c} 2^{u_1}z_1 & \mathbb{A};C_1,\mathbf{A},A_2:A \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{u_r}z_r & \mathbb{B};B_1,\mathbf{B},D_1:B \end{array}\right) \tag{4.1}$$

where

$$C_1=(1-k;v_1,\cdots,v_r;1),(-\rho-\alpha+\beta-k;u_1,\cdots,u_r;1),(-\rho-\alpha-k;u_1,\cdots,u_r;1) \tag{4.2}$$

$$D_1=(1;v_1,\cdots,v_r;1),(n-\rho-k;u_1,\cdots,u_r;1),(-1-\rho-\beta-k-\alpha-n;u_1,\cdots,u_r;1) \tag{4.3}$$

provided

$$Re(\alpha+\beta+1), u_i, v_i > 0 (i=1,\cdots,r),\ Re(1+\lambda) - \sum_{i=1}^{r} v_i \min_{\substack{1\leqslant j\leqslant m_i\\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}} + D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0.$$

$$Re(1+\rho+\alpha-\beta+k) + \sum_{i=1}^{r} u_i \min_{\substack{1\leqslant j\leqslant m_i\\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}} + D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0 \text{ and}$$

Proof

To prove (4.1), let

$$x^{\lambda}(1-x)^{\rho}\daleth\left(z_1\frac{(1-x)^{u_1}}{x^{v_1}},\cdots,z_r\frac{(1-x)^{u_r}}{x^{v_r}}\right) = \sum_{R=0}^{\infty} C_R P_R^{(\alpha,\beta)}(x) \tag{4.4}$$

The above equation is valid since the expression on the left-hand side is continuous and bounded variation in the interval $(-1,1)$. Multiplying both sides of the equation (4.4) by $(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)$ and integrating with respect to x from 1 to 1.

Theorem 6.

$$x^{\lambda}(1+x)^{\sigma}\daleth\left(z_1 x^{u_1}(1+x)^{v_1},\cdots,z_r x^{u_r}(1+x)^{v_r}\right) = \sum_{n,k=0}^{\infty}\frac{(-)^n 2^{k+\sigma}(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)}{k!\Gamma(1+\alpha+n)}P_n^{(\alpha,\beta)}(x)$$

$$\daleth^{U;m_r+1,n_r+2:V}_{X;p_{i_r}+3,q_{i_r}+3,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} 2^{v_1}z_1 & \mathbb{A};(-k+\beta-\sigma-\alpha;v_1,\cdots,v_r;1),(-k-\beta-\sigma;v_1,\cdots,v_r;1),\mathbf{A},(-\lambda;u_1,\cdots,u_r;1):A\\ \cdot & \cdot\\ \cdot & \cdot\\ 2^{v_r}z_r & \mathbb{B};(-\lambda+k;u_1,\cdots,u_r;1),\mathbf{B},(n-k;v_1,\cdots,v_r;1),(-1-n-\alpha-\beta-\sigma-k;v_1,\cdots,v_r;1):B\end{array}\right) \tag{4.5}$$

provided

$$Re(\alpha+\beta+1), u_i, v_i > 0 (i=1,\cdots,r),\ Re(1+\lambda) + \sum_{i=1}^{r} u_i \min_{\substack{1\leqslant j\leqslant m_i\\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}} + D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0.$$

$$Re(1-k+\alpha+\beta+\sigma) + \sum_{i=1}^{r} v_i \min_{\substack{1\leqslant j\leqslant m_i\\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}} + D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0$$

Theorem 7.

$$x^{\lambda}(1-x)^{\rho}\daleth\left(z_1 x^{u_1}(1-x)^{v_1},\cdots,z_r x^{u_r}(1-x)^{v_r}\right) = \sum_{n,k=0}^{\infty}\frac{(-)^n 2^{k+\rho}(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)}{k!\Gamma(1+\alpha+n)}P_n^{(\alpha,\beta)}(x)$$

$$\daleth^{U;m_r+1,n_r+2:V}_{X;p_{i_r}+3,q_{i_r}+3,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} 2^{v_1}z_1 & \mathbb{A};(\beta-\rho-\alpha-k;v_1,\cdots,v_r;1),(-k-\rho-\alpha;v_1,\cdots,v_r;1),\mathbf{A},(-\lambda;u_1,\cdots,u_r;1):A\\ \cdot & \cdot\\ \cdot & \cdot\\ 2^{v_r}z_r & \mathbb{B};(-\lambda+k;u_1,\cdots,u_r;1),\mathbf{B},(n-k;v_1,\cdots,v_r;1),(-1-n-\alpha-\beta-\rho-k;v_1,\cdots,v_r;1):B\end{array}\right) \tag{4.6}$$

provided

$$Re(\alpha+\beta+1), u_i, v_i > 0 (i=1,\cdots,r),\ Re(1+\lambda) + \sum_{i=1}^{r} u_i \min_{\substack{1\leqslant j\leqslant m_i\\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}} + D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0.$$

$$Re(1+k+\alpha-\beta+\rho)+\sum_{i=1}^{r}v_i \min_{\substack{1\leqslant j\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0$$

Theorem 8.

$$\int_{-1}^{1}x^{\lambda}(1-x)^{\rho}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)\gimel\left(z_1\frac{(1-x)^{u_1}}{x^{v_1}},\cdots,z_r\frac{(1-x)^{u_r}}{x^{v_r}}\right)\mathrm{d}x=$$

$$\frac{(-)^n2^{\rho+\beta+1}\Gamma(1+\beta+n)}{n!}\sum_{k=0}^{\infty}\frac{2^k}{k!}\gimel^{U;m_r+2,n_r+3:V}_{X;p_{i_r}+5,q_{i_r}+5,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c}2^{u_1}z_1 & \mathbb{A};C_2,\mathbf{A},A_2:A\\ \cdot & \cdot\\ \cdot & \cdot\\ 2^{u_r}z_r & \mathbb{B};B_1,\mathbf{B},D_2:B\end{array}\right) \tag{4.7}$$

where

$$C_2=(1-k;v_1,\cdots,v_r;1),(-\sigma-k;u_1,\cdots,u_r;1),(-\sigma-\beta-k;u_1,\cdots,u_r;1) \tag{4.8}$$

$$D_2=(1;v_1,\cdots,v_r;1),(n-\sigma-k;u_1,\cdots,u_r;1),(-1-\sigma-k-\alpha-\beta-n;u_1,\cdots,u_r;1) \tag{4.9}$$

provided

$$Re(1+\alpha+\beta),u_i,v_i>0(i=1,\cdots,r),\ Re(1+\lambda)-\sum_{i=1}^{r}v_i \min_{\substack{1\leqslant j\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0.$$

$$Re(1+\sigma+\beta+k)+\sum_{i=1}^{r}u_i \min_{\substack{1\leqslant j\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0$$

5. Conclusion.

The importance of our all the results lies in their manifold generality. Firstly, in view of Jacobi polynomials making use of special cases, they can be reduced to a large number of formulae involving simpler special functions (Ultraspherical polynomials, Chebyshev, Legendre, Bateman's, Hermite, Laguerre polynomials and others). Secondly by specialising the various parameters as well as variables in the generalized multivariable Gimel-function, we get a several formulae involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several intersting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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