Fixed Point Theorems in Dislocated Quasi-b-Metric Space for Two Self Maps

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Abstract*:The purpose of this paper is to establish a common fixed point result for two self-maps, using generalised contraction condition in dislocated quasi-b-metric space. This result is an extension and generalisation of the result of Mujeeb Ur Rahman[8].*

Keywords—*fixed point, b-metric spaces, quasi-b-metric spaces.*

I. Introduction

Fixed point theory is one of the most dynamic research subjects in non-linear analysis. In 1912 Brouwer [3] proved a result that a unit closed ball in R^n has a fixed point. The most remarkable result in fixed point theory was given by Banach[2] in 1922. He proved that each contraction in a complete metric space has a unique fixed point. Later on, many authors generalized the Banach fixed point theorem in various ways [6,7,10,11,12,14]. P. Hitzler and A.K.Seda generalized, the well-knownBanach Contraction Principle of metric space to the dislocated metric space. This result played a key role in the development of logic programming semantics [5].

The quasi metric spaces were introduced by Wilson [13]in 1931 as a generalisation of metric spaces. In 1989 Bakhtin [1] introduced the notion of b-metric space as generalisation of metric space. Klin-eamet.al [4] introduced dislocated quasi-b-metric space which generalize quasi-b-metric space and b-metric like space.\\

In this present paper we prove a common fixed-point theorem in dislocated quasi b-metric space for two self-maps. This resultextends and generalise the result of Mujeeb Ur Rahman [8]and many results in literature.

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II. Preliminaries

The following definitions are necessary to prove our result, which is found in [9].

Definition 2.1. Let **X** be a non-empty set, $k \geq 1$ be a given real number and $d: X \times X \rightarrow R_+$ be a function. Is called dislocated quasi-b-metric for all x, y and z in X if the following conditions are satisfied:

1. $d(x, y) = 0 = d(y, x)$ if and only if $x = y$.
2. $d(x, z) \leq k[d(x, y) + d(y, z)]$

A pair (X, d) is called a dislocated quasi-b-metric space or dq-b-metric space. If k =1, then dq-b-metric reduces to dislocated quasi metric space.

Remark: If k = 1 the dislocated quasi-b-metric space reduce to dislocated quasi metricspace.Therefore, every dislocated quasi metric space is dislocated quasi-b-metric space andevery b-metric space is dislocated quasi-b-metric space with same coefficient k and zero self-distance, but converse is not true.

Example: Let X = R and suppose $d(x, y) = |2x-y|^2 + |2x+y|^2$ Then (X, d) is a dislocatedquasi-b-metric space with the coefficient k = 2. But it is neither dislocated quasi-metricspace nor b-metric space.

Definition 2.2. The sequence $\{x_n\}$ is called dq-b-convergent sequence in (X, d) if forn $\geq$ N we have $d(x_n, x) < \varepsilon$ where $\varepsilon > 0$ then x is called the dq-b-limit of sequence $\{x_n.\}$.

Definition 2.3. Let (X,d) be a dq-b-metric space. The sequence $\{x_n\}$in X is called aCauchy sequence if and only if for all $\varepsilon > 0$ there exist $n_0 \in N$ such that for each m, $n \geq n_0$we have $d(x_n, x_m) < \varepsilon$.

Definition 2.4. A dq-b-metric space (X,d) is said to be complete if every Cauchy sequence in X converges to a point of X.

The following results are from [9].

Lemma 2.5. Limit of a convergent sequence in dislocated quasi-b-metric space is unique.

Lemma 2.6. Let (X,d) be a dislocated quasi-b-metric space and $\{x_n\}$be a sequence indq-b-metric space such that $d(x_n, x_{n+1}) \leq d(x_{n-1},x_n)$ for n = 1, 2, 3,… and $0 \leq \alpha k \leq 1, \alpha \in [0, 1)$ and k is defined in dq-b-metric space. Then$\{x_n\}$is a Cauchy sequence in X.

III.MAIN RESULT

Theorem 3.1. Let (X,d) be a complete dq-b-metric space with $k \geq 1$ and S and T aretwo self-mappings S,T: $X \to X$ satisfying the condition

$$d(Sx,Ty) \leq h \max\{ d(x, y), d(x,Sx), d(y, Ty), d(x, Ty), d(y,Sx)\}$$

for all $x,y \in X$ with $2hk < 1$ and $0 \leq h \leq 1$ then S and T have unique common fixedpoint.

Proof. Let x_0 be arbitrary point in X and we define $\{x_n\}$ in X as

$$x_{n+1} = Sx_n \text{ and } x_{n+2} = Tx_{n+1} \quad (1)$$

for n = 0, 1, 2, …

Consider

$d(x_{n+1}, x_{n+2}) = d(Sx_n, Tx_{n+1})$

$\leq h \max\{d(x_n, x_{n+1}), d(x_n, Sx_n), d(x_{n+1},Tx_{n+1}),d(x_n, Tx_{n+1}), d(x_{n+1},Sx_n)\}$,

$\leq h \max\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1},x_{n+2}),d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1})\}$,

$\leq h \max\{d(x_n, x_{n+1}), d(x_{n+1},x_{n+2}),d(x_n, x_{n+2})\}$.

now different cases arise.

Case 1.If $\max \{d (x_n ,x_{n+1}),d(x_{n+1},x_{n+2}),d(x_n,x_{n+2})\} =d(x_n,x_{n+1})$

then $d(x_{n+1} ,x_{n+2}) \leq hd(x_n,x_{n+1})$.

similarly, $d(x_n,x_{n+1}) \leq hd(x_{n-1},x_n)$

Continuing this process, we get $d(x_{n+1},x_{n+2}) \leq h^{n+1}d(x_0,x_1)$.

Case 2. If $\max\{d(x_n , x_{n+1}),d(x_{n+1} , x_{n+2}), d(x_n, x_{n+2})\} =d(x_{n+1} , x_{n+2})$

Then$d(x_{n+1},x_{n+2}) \leq h[d(x_{n+1},x_{n+2})]$,

$(1 - h)d(x_{n+1},x_{n+2}) \leq 0$, this implies $d(x_{n+1},x_{n+2}) = 0$.

Case 3. If $\max \{d(x_n , x_{n+1}),d(x_{n+1} , x_{n+2}), d(x_n, x_{n+2})\} = d(x_n,x_{n+2})$

Then$d(x_{n+1}, x_{n+2}) \leq h[d(x_{n+1},x_{n+2})]$,

$\leq hk[d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2})]$,

$\leq \frac{hk}{1-hk}[d(x_n,x_{n+1})]$,

Where$\alpha = \frac{hk}{1-hk} < 1$

Continuing the process, we get $d(x_{n+1},x_{n+2}) \leq \alpha^{n+1}d(x_0,x_1)$.

Using lemma 2.6 we get $\{x_n\}$ is Cauchy sequence in complete dq-b-metric space. So there exists $u \in X$ such that $\lim_{n\to\infty}x_n= u$.

Now we show that u is a fixed point of T.

Consider

$d(u,Tu) \leq kd(u,x_{n+1}) + d(x_{n+1},Tu)$,

$\leq kd(u,x_{n+1}) + kd(Sx_n,Tu)$,

$\leq kd(u,x_{n+1}) + k[h\max\{d(x_n,u),d(x_n,Sx_n),d(u,Tu),d(x_n,Tu),d(u,Sx_n)\}]$,

$\leq kd(u,x_{n+1}) + k[h\max\{d(x_n,u),d(x_n,x_{n+1}),d(u,Tu),d(x_n,Tu),d(u,x_{n+1})\}]$,

now different cases arise.

Case 1. If $\max\{d(x_n, u), d(x_n, x_{n+1}), d(u,Tu), d(x_n,Tu), d(u, x_{n+1})\} = d(x_n, u)$ then

$d(u,Tu) \leq kd(u, x_{n+1}) + khd(x_n,u)$

as $n \to \infty$, $d(u,Tu) = 0$.

Therefore, u is a fixed point of T.

Case 2. Ifmax $\{d(x_n, u), d(x_n, x_{n+1}), d(u, Tu), d(x_n,Tu), d(u,x_{n+1})\} = d(x_n,x_{n+1})$ then

$d(u,Tu) \leq kd(u, x_{n+1}) + khd(x_n,x_{n+1})$

as $n \to \infty$, $d(u,Tu) = 0$.

Therefore, u is a fixed point of T.

Case 3. Ifmax $\{d(x_n, u), d(x_n, x_{n+1}), d(u,Tu), d(x_n,Tu), d(u,x_{n+1})\} = d(x_n,Tu)$ then

$d(u,Tu) \leq kd(u,x_{n+1}) + khd(x_n,Tu)$

as $n \to \infty$, $(1 - kh)d(u,Tu) = 0$

implies $d(u,Tu) = 0$therefore is a fixed point of T.

Case 4. Ifmax $\{d(x_n,u),d(x_n,x_{n+1}),d(u,Tu),d(x_n,Tu),d(u,x_{n+1})\} = d(u,Tu)$ then

$d(u,Tu), \leq kd(u,x_{n+1}) + khd(u, Tu)$

$d(u,Tu), \leq \frac{hk}{1-hk} d(u,x_{n+1})$

as $n \to \infty$, $d(u,Tu) = 0$

Therefore, u is a fixed point of T.

Case 5. If max $\{d(x_n,u),d(x_n,x_{n+1}),d(u,Tu),d(x_n,Tu),d(u,x_{n+1})\} = d(u,x_{n+1})$ then

$d(u,Tu) \leq kd(u,x_{n+1}) + khd(u,x_{n+1})$,

$\leq k(1 + h)d(u,x_{n+1})$

as $d(u,Tu) = 0$ then u is a fixedpointof T.

Nowwe show that u is fixed point of S.

Consider

$d(u,Su) \leq kd(u,x_{n+2}) + d(x_{n+2},Su)$,

$\leq kd(u,x_{n+2}) + kd(Su,Tx_{n+1})$,

$\leq kd(u,x_{n+2}) + k[hmax\{d(u,x_{n+1}),d(x_{n+1},Tx_{n+1}),d(u,Su),d(x_{n+1},Su),d(u,Tx_{n+1})\}]$,

$\leq kd(u,x_{n+2}) + k[hmax\{d(u,x_{n+1}),d(x_{n+1},x_{n+2}),d(u,Su),d(x_{n+1},Su),d(u,x_{n+2})\}]$

Now different cases arise

Case 1. Ifmax $\{d(u,x_{n+1}),d(x_{n+1},x_{n+2}),d(u,Su),d(x_{n+1},Su),d(u,x_{n+2})\}] = d(u,x_{n+1})$ then

$d(u,Su) \leq kd(u,x_{n+2}) + khd(u,x_{n+1})$

as $n \to \infty$, $d(u,Su) = 0$.therefore u is a fixed point of S.

Case 2. If max $\{d(u,x_{n+1}),d(x_{n+1},x_{n+2}),d(u,Su),d(x_{n+1},Su),d(u,x_{n+2})\}] = d(u,Su)$ then

$d(u, Su) \leq kd(u, x_{n+2}) + khd(u, Su)$,

$\leq \frac{k}{1-kh} d(u, x_{n+2})$

as $n \to \infty$, $d(u,Su) = 0$. then u is a fixed point of S.

Case 3 If max $\{d(u, x_{n+1}), d(x_{n+1}, x_{n+2}), d(u, Su), d(x_{n+1}, Su), d(u, x_{n+2})\}] = d(x_{n+1}, x_{n+2})$ then

$d(u, Su) \leq kd(u, x_{n+2}) + khd(x_{n+1}, x_{n+2})$,

as $n \to \infty$, $d(u, Su) = 0$. then u is a fixed point of S.

Case 4. If max$\{d(u, x_{n+1}), d(x_{n+1}, x_{n+2}), d(u,Su), d(x_{n+1}, Su), d(u,x_{n+2})\}] = d(u, x_{n+2})$ then

$d(u, Su) \leq kd(u, x_{n+2}) + khd(u, x_{n+2})$,

$\leq k(1 + h)d(u, x_{n+2})$

as $n \to \infty$, $d(u, Su) = 0$. then u is a fixed point of S.

Case 5. If max$\{d(u, x_{n+1}), d(x_{n+1}, x_{n+2}), d(u, Su), d(x_{n+1}, Su), d(u, x_{n+2})\}] = d(x_{n+1}, Su)$ then

$d(u, Su) \leq kd(u, x_{n+2}) + khd(x_{n+1}, Su)$,

$\leq kd(u, x_{n+2}) + k^2h[d(x_{n+1,} u), +d(u, Su)]$

$\leq \frac{k}{1-kh}d(u, x_{n+2})+\frac{k^2h}{1-k^2h}d(x_{n+1,} u),$

as $n \to \infty$, $d(u, Su) = 0$. then u is a fixed point of S.

So u a common fixed point of S and T.

Uniqueness

Now we show that u is unique fixed point of S and T. let u and v be two different fixed point of S and T.

i.e. $Su= u = Tu$, and $Sv= v = Tv$

Consider $d(u,v) =d(Su,Tv)$.

$\leq hmax\{d(u,v),d(u,Su),d(v,Tv),d(u, Tv),d(v,Su)\}$

$\leq hmax\{d(u,v),d(u,u),d(v,v),d(u,v),d(v,u)\}$

$\leq hmax\{d(u,v),d(v,u)\}$

$\leq hd(u,v)$

$(1 - h)d(u,v) \leq 0$

$d(u,v) = 0$ implies $u = v$

Therefore, u is a unique common fixed point of S and T.

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