Triple Series Relations Involving Multivariable Gimel-Function

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ABSTRACT.
Agrawal [5] given triple series relations about the H-function of two variables. In this paper, we have established two triple infinite series relations concerning the multivariable Gimel-function defined here. On specialization of the parameters, a number of interesting triple, double and single series relations involving simpler functions of one or several variables that occur rather frequently in problems of Applied Mathematics and Mathematical physics problems can be deduced as particular cases from our results.

KEYWORDS : Generalized multivariable Gimel-function, multiple integral contours, double series relations.

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1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C},\mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. We define a generalized transcendental function of several complex variables.

We define a generalized transcendental function of several complex variables noted $\gimel$.

$$\gimel(z_1,\cdots,z_r)=\gimel^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)};n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2}:R_2;p_{i_3},q_{i_3},\tau_{i_3}:R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|}\mathrm{z}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\end{array}\right.$$

$$\begin{array}{c}[(\mathrm{a}_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}};[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},\\ [\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{1,q_{i_2}};\end{array}$$

$$\begin{array}{l}[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],\\ [\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{1,q_{i_3}};\cdots;\quad\cdots\end{array}$$

$$\begin{array}{l}[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{n_r+1,p_r}]:\ [(\mathrm{c}^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]\\ [\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{1,q_r}]:\ [(\mathrm{d}^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]\end{array}$$

$$\left.\begin{array}{l};\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})_{m^{(r)}+1,p^{(r)}_i}]\\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})_{n^{(r)}+1,q^{(r)}_i}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha^{(k)}_{2j}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha^{(k)}_{2ji_2}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta^{(k)}_{2ji2}s_k)]}$$

$$\frac{\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)}]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $n_2,\cdots,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2,\cdots,0\leqslant m_r,0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}},$

$0\leqslant n^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant p_{i^{(r)}}.$

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r).$

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r).$

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r).$

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k).$

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k).$

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k).$

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r).$

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r).$

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n^{(k)});(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m^{(k)});(k=1,\cdots,r).$

$a_{kji_k}\in\mathbb{C};(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r).$

$b_{kji_k}\in\mathbb{C};(j=1,\cdots,q_{i_k});(k=2,\cdots,r).$

$d^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = m^{(k)} + 1, \cdots, q_{i^{(k)}}); (k = 1, \cdots, r).$

$\gamma^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r).$

The contour L_k is in the $s_k(k = 1, \cdots, r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1 - a_{2j} + \sum_{k=1}^{2} \alpha^{(k)}_{2j} s_k\right) (j = 1, \cdots, n_2), \Gamma^{A_{3j}}\left(1 - a_{3j} + \sum_{k=1}^{3} \alpha^{(k)}_{3j} s_k\right)$ $(j = 1, \cdots, n_3), \cdots, \Gamma^{A_{rj}}\left(1 - a_{rj} + \sum_{i=1}^{r} \alpha^{(i)}_{rj}\right) (j = 1, \cdots, n_r), \Gamma^{C^{(k)}_j}\left(1 - c^{(k)}_j + \gamma^{(k)}_j s_k\right) (j = 1, \cdots, n^{(k)})(k = 1, \cdots, r)$to the right of the contour L_k and the poles of $\Gamma^{D^{(k)}_j}\left(d^{(k)}_j - \delta^{(k)}_j s_k\right) (j = 1, \cdots, m^{(k)})(k = 1, \cdots, r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2} A^{(k)}_i \pi$ where

$$A^{(k)}_i = \sum_{j=1}^{m^{(k)}} D^{(k)}_j \delta^{(k)}_j + \sum_{j=1}^{n^{(k)}} C^{(k)}_j \gamma^{(k)}_j - \tau_{i^{(k)}} \left(\sum_{j=m^{(k)}+1}^{q^{(k)}_i} D^{(k)}_{ji^{(k)}} \delta^{(k)}_{ji^{(k)}} + \sum_{j=n^{(k)}+1}^{p^{(k)}_i} C^{(k)}_{ji^{(k)}} \gamma^{(k)}_{ji^{(k)}} \right) +$$

$$-\tau_{i_2} \left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2} \alpha^{(k)}_{2ji_2} + \sum_{j=1}^{q_{i_2}} B_{2ji_2} \beta^{(k)}_{2ji_2} \right) - \cdots - \tau_{i_r} \left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r} \alpha^{(k)}_{rji_r} + \sum_{j=1}^{q_{i_r}} B_{rji_r} \beta^{(k)}_{rji_r} \right) \tag{1.4}$$

Following the lines of Braaksma ([3] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \rightarrow 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \rightarrow \infty$ where $i = 1, \cdots, r$:

$\alpha_i = \min_{1 \leqslant j \leqslant m^{(i)}} Re \left[D^{(i)}_j \left(\frac{d^{(i)}_j}{\delta^{(i)}_j} \right) \right]$ and $\beta_i = \max_{1 \leqslant j \leqslant n^{(i)}} Re \left[C^{(i)}_j \left(\frac{c^{(i)}_j - 1}{\gamma^{(i)}_j} \right) \right]$

Remark 1.
If $n_2 = \cdots = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$ $A_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the multivariable Gimel-function reduces in the multivariable Aleph- function defined by Ayant [2].

Remark 2.
If $n_2 = \cdots = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [6].

Remark 3.
If$A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)}$ $= \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [5].

Remark 4.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H-function defined by Srivastava and panda [7,8].

In your investigation, we shall use the following notations.

$$\mathbb{A} =[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$
$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \quad (1.5)$$

$$\mathbf{A} = [(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \quad (1.6)$$

$$A = [(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \quad (1.7)$$

$$\mathbb{B} = [\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})_{1,q_{i_{r-1}}}] \quad (1.8)$$

$$\mathbf{B} = [\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{1,q_{i_r}}] \quad (1.9)$$

$$\mathrm{B} = [(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}];\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \quad (1.10)$$

$$U = 0,n_2;0,n_3;\cdots;0,n_{r-1};V = m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \quad (1.11)$$

$$X = p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y = p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \quad (1.12)$$

2. Triple-series relations.

In this section, we establish two triple-series relations :

Theorem 1.

$$\sum_{u,v,w=0}^{\infty}\frac{(a)_{u+v+w}(1-a)_{u+v+w}}{u!v!w!}\,\gimel_{X;p_{i_r}+3,q_{i_r}+2,\tau_{i_r}:R_r:Y}^{U;0,n_r+3:V}\left(\begin{array}{c}\mathrm{a}_1\\ \cdot\\ \cdot\\ \mathrm{a}_r\end{array}\right|$$

$$\left.\begin{array}{c}\mathbb{A};(\text{l-b-u};\mathrm{h}_1,\cdots,h_r;1),(l-c-v;k_1,\cdots,k_r;1),(l-d-w;l_1,\cdots,l_r;1),\mathbf{A}:A\\ \cdot\\ \cdot\\ \mathbb{B};\mathbf{B},(\text{l-e-u-v-w};2\mathrm{H}_1,\cdots,2H_r;1),(e-2b-2c-2d-u-v-w;2(h_1+k_1+l_1-H_1),\cdots,2(h_r+k_r+l_r-K_r);1):B\end{array}\right)$$

$$= 2^{1-2(b+c+d)}\pi\,\gimel_{X;p_{i_r}+3,q_{i_r}+4,\tau_{i_r}:R_r:Y}^{U;0,n_r+3:V}\left(\begin{array}{c}4^{-(h_1+k_1+l_1)}a_1\\ \cdot\\ \cdot\\ 4^{-(h_r+k_r+l_r)}a_r\end{array}\right|\begin{array}{c}\mathbb{A};(\text{1-b};\mathrm{h}_1,\cdots,h_r;1),\\ \cdot\\ \cdot\\ \mathbb{B};\mathbf{B},(1-\frac{a+e}{2};H_1,\cdots,H_r;1),(\frac{1+a-e}{2};H_1,\cdots,H_r;1),\end{array}$$

$$\begin{pmatrix} \text{(1-c;k}_1,\cdots,k_r;1),(1-d;l_1,\cdots,l_r;1),\mathbf{A}:A \\ \cdot \\ \cdot \\ \left(\frac{1-a+e}{2}-b-c-d;h_1+k_1+l_1-H_1,\cdots,h_r+k_r+l_r-H_r;1\right),\left(\frac{a+e}{2}-b-c-d;h_1+k_1+l_1-H_1,\cdots,h_r+k_r+l_r-H_r;1\right) \end{pmatrix} \quad (2.1)$$

Provided

$h_i,k_i,l_i,H_i>0(i=1,\cdots,r)$ and verify $h_i+k_i+l_i-H_i\geqslant 0$. The triple serie involved in (2.1) is assumed to be absolutely convergent and $|arg(a_i)|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Proof

To establish (2.1), expressing the multivariable Gimel-function on the left-hand side of (2.1) in terms of Mellin-Barnes multiple integrals contour with the help of (1.1), changing the order of integrations and summations (as the triple series involved is absolutely convergent) and then evaluating the inner series with the help of the following result of Parihar's formula ([4], p. 216, (2.2)).

$$\sum_{u,v,w=0}^{\infty}\frac{(a)_{u+v+w}(1-a)_{u+v+w}(b)_u(c)_v(d)_w}{(e)_{u+v+w}(1+2b+2c+2d-e)_{u+v+w}u!v!w!}=$$

$$\frac{2^{1-2(b+c+d)}\pi\Gamma(e)\Gamma\left(1+\frac{b+c+d}{2}-e\right)}{\Gamma\left(\frac{e+a}{2}\right)\Gamma\left(\frac{1+e-a}{2}\right)\Gamma\left(\frac{1+a-e}{2}+b+c+d\right)\Gamma\left(1-\frac{a-e}{2}+b+c+d\right)} \quad (2.2)$$

and interpreting the Mellin-Barnes multiple integrals contour in terms of the multivariable Gimel-function with the help of (1.1) , we get the desired result (2.1).

Theorem 2.

$$\sum_{u,v,w=0}^{\infty}\frac{1}{u!v!w!}\gimel^{U;0,n_r+5:V}_{X;p_{i_r}+5,q_{i_r}+2,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} \text{a}_1 & \mathbb{A};\text{(1-a-u-v-w;2H}_1,\cdots,2H_r;1),(1-b-u-v-w;2K_1,\cdots,2K_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ \text{a}_r & \mathbb{B};\mathbf{B},\left(\frac{1-a-b}{2}-u-v-w;H_1+K_1,\cdots,H_r+K_r;1\right), \end{array}\right.$$

$$\left.\begin{array}{c} \text{(l-c-u;h}_1,\cdots,h_r;1),(l-d-v;k_1,\cdots,k_r;1),(l-e-w;l_1,\cdots,l_r;1),\mathbf{A}:A \\ \cdot \\ \cdot \\ \text{(1-2c-2d-2e-u-v-w;2(h}_1+k_1+l_1),\cdots,2(h_r+k_r+l_r);1):B \end{array}\right)=2^{1-2(b+c+d)}\pi\,\gimel^{U;0,n_r+6:V}_{X;p_{i_r}+6,q_{i_r}+5,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} 4^{-(h_1+k_1+l_1)}a_1 & \mathbb{A};\text{(1-a;2H}_1,\cdots,2H_r;1),(1-b;2K_1,\cdots,2K_r;1),(1-c;h_1,\cdots,h_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ 4^{-(h_r+k_r+l_r)}a_r & \mathbb{B};\mathbf{B},\left(\frac{1-a}{2};H_1,\cdots,H_r;1\right),\left(\frac{1-b}{2};K_1,\cdots,K_r;1\right),(1-c-d-e;h_1+k_1+l_1,\cdots,h_r+k_r+l_r;1), \end{array}\right.$$

$$\left.\begin{array}{c} \text{(1-d;k}_1,\cdots,k_r;1),(1-e;l_1,\cdots,l_r;1),\left(\frac{1+a+b}{2}-c-d-e;h_1+k_1+l_1-H_1-K_1,\cdots,h_r+k_r+l_r-H_r-K_r\right),\mathbf{A}:A \\ \cdot \\ \cdot \\ \left(\frac{1+a}{2}-c-d-e;h_1+k_1+l_1-H_1,\cdots,h_r+k_r+l_r-H_r;1\right),\left(\frac{1+b}{2}-c-d-e;h_1+k_1+l_1-K_1,\cdots,h_r+k_r+l_r-K_r;1\right) \end{array}\right) \quad (2.3)$$

Provided

$h_i,k_i,l_i,H_i,K_i>0(i=1,\cdots,r)$ and verify $h_i+k_i+l_i-H_i-K_i\geqslant 0$. The triple serie involved in (2.3) is assumed to be absolutely convergent and $|arg(a_i)|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

The proof of (2.3) can be developed on the same lines as suggested with (2.1). Here we use another Parihar's formula ([4], p. 216, (2.1)).

3. Special cases.

Taking $h_i = 0 (i = 1, \cdots, r), b \to 0$ in (2.1), we shall obtain the following double series relation :

Corollary 1.

$$\sum_{v,w=0}^{\infty} \frac{(a)_{v+w}(1-a)_{v+w}}{v!w!} \gimel^{U;0,n_r+2:V}_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y} \left(\begin{array}{c|c} \mathrm{a}_1 & \mathbb{A};\ (\text{l-c-v};\mathrm{k}_1,\cdots,k_r;1),(l-d-w;l_1,\cdots,l_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{a}_r & \mathbb{B};\ \mathbf{B},(\text{l-e-v-w};2\mathrm{H}_1,\cdots,2H_r;1),(e-2c-2d-v-w;2(k_1+l_1-H_1),\cdots,2(k_r+l_r-K_r);1):B \end{array} \right)$$

$$= 2^{1-2(c+d)}\pi\ \gimel^{U;0,n_r+2:V}_{X;p_{i_r}+2,q_{i_r}+4,\tau_{i_r}:R_r:Y} \left(\begin{array}{c|c} 4^{-(k_1+l_1)}a_1 & \mathbb{A};(\text{1-c};\mathrm{k}_1,\cdots,k_r;1),(\text{1-d};\mathrm{l}_1,\cdots,l_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ 4^{-(k_r+l_r)}a_r & \mathbb{B};\mathbf{B},(1-\frac{a+e}{2};H_1,\cdots,H_r;1),(\frac{1+a-e}{2};H_1,\cdots,H_r;1),(\frac{1-a+e}{2}-c-d;k_1+l_1-H_1,\cdots,k_r+l_r-H_r;1),(\frac{a+e}{2}-c-d;k_1+l_1-H_1,\cdots,k_r+l_r-H_r;1) \end{array} \right) \tag{3.1}$$

Provided

$k_i, l_i, H_i > 0 (i = 1, \cdots, r)$ and verify $k_i + l_i - H_i \geqslant 0$. The triple serie involved in (3.1) is assumed to be absolutely convergent and $|arg(a_i)| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Taking $k_i = 0 (i = 1, \cdots, r), c \to 0$ in (3.1), we obtain the single series relations :

Corollary 2.

$$\sum_{w=0}^{\infty} \frac{(a)_{w}(1-a)_{w}}{w!} \gimel^{U;0,n_r+1:V}_{X;p_{i_r}+1,q_{i_r}+2,\tau_{i_r}:R_r:Y} \left(\begin{array}{c|c} \mathrm{a}_1 & \mathbb{A};\ (\text{l-d-w};\mathrm{l}_1,\cdots,l_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{a}_r & \mathbb{B};\ \mathbf{B},(\text{l-e-v-w};2\mathrm{H}_1,\cdots,2H_r;1),(e-2d-w;2(l_1-H_1),\cdots,2(l_r-K_r);1):B \end{array} \right)$$

$$= 2^{1-2d}\pi\ \gimel^{U;0,n_r+1:V}_{X;p_{i_r}+1,q_{i_r}+4,\tau_{i_r}:R_r:Y} \left(\begin{array}{c|c} 4^{-l_1}a_1 & \mathbb{A};(\text{1-d};\mathrm{l}_1,\cdots,l_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ 4^{-l_r}a_r & \mathbb{B};\mathbf{B},(1-\frac{a+e}{2};H_1,\cdots,H_r;1),(\frac{1+a-e}{2};H_1,\cdots,H_r;1),(\frac{1-a+e}{2}-d;l_1-H_1,\cdots,l_r-H_r;1),(\frac{a+e}{2}-d;l_1-H_1,\cdots,l_r-H_r;1) \end{array} \right) \tag{3.2}$$

Provided

$l_i, H_i > 0 (i = 1, \cdots, r)$ and verify $l_i - H_i \geqslant 0$. The triple serie involved in (3.1) is assumed to be absolutely convergent and $|arg(a_i)| < \frac{1}{2} A_i^{(k)} \pi$ where $A_i^{(k)}$ is defined by (1.4).

Similar type of double and single-series relations can be obtained from (2.3).

4. Conclusion.

The main triples-series relations (2.1) and (2.3) established here are unified and act as key formulae. Thus the multivariable Gimel-function occurring in these triple-series relations can be suitably specialized to a remarkably wide variety of special functions (or product of several such special functions) which are expressible in terms of E, F, G, H, I, Aleph-functions of one or more variables.

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