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# Composition Operators on Weighted Orlicz Sequence Spaces

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**Abstract.** In this paper, we study the boundedness of composition operators between any two weighted Orlicz sequence spaces

**Keywords.** Composition operators, Boundedness, weighted Orlicz sequence spaces.

## 1 Introduction

Let $\phi : [0,\ \infty) \to [0,\ \infty)$ be a Young function, that is, a nondecreasing continuous convex function for which $\phi(0) = 0$ and $\lim_{x\to\infty} \phi(x) = \infty$. Let $(X, \Sigma, \mu)$ be a $\sigma$-finite and purely atomic measure space, that is

$$X = \bigcup_{n=1}^{\infty} A_n,$$

where $A_n$ are atoms with the measure $\mu(A_n) = a_n > 0$ for all $n \in \mathbb{N}$ and $w = \{w_n\}$ be a weight in $X$ i.e positive and summable real valued sequence. Then the *weighted Orlicz sequence space* $l_w^{\phi}(\{a_n\})$ is defined as the space of all real sequences $\mathsf{f} = \{\mathsf{f}_n\}_{n=1}^{\infty}$ such that $I_{\phi,w}(\lambda\mathsf{f}, \{a_n\}) < \infty$ for some $\lambda > 0$, where

$$I_{\phi,w}(\mathsf{f}, \{a_n\}) = \sum_{n=1}^{\infty} \phi(|\mathsf{f}_n|) w_n a_n.$$

This space is a Banach space with the norm

$$\|\mathsf{f}\|_{\phi,w,\{a_n\}} = \inf\{\lambda > 0 \mid I_{\phi,w}(\mathsf{f}/\lambda, \{a_n\}) \le 1\}.$$

Throughout the paper, we assume $(X, \Sigma, \mu)$ to be a $\sigma$-finite and purely atomic measure space with atoms $\{A_n\}$ of measure $\mu(A_n) = a_n > 0$ for any $n \in \mathbb{N}, \tau : X \to X$ to be a measurable non-singular transformation such that $\tau(X) = X$ and $b_n := \mu(\tau^{-1}(A_n))/\mu(A_n)$.

Composition operators on Orlicz spaces have also been studied in [3], [4], [5],[9] and [17]. The techniques used in this paper essentially depend on the conditions of embedding of one Orlicz space into another (see, [13, Page 48] and [19] for details).

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# 2 Boundedness of Composition Operators

In this section, we study the boundedness of composition operators on weighted Orlicz sequence spaces.

**Theorem 2.1.** *Let $w_1 = \{w_{1,n}\}$ and $w_2 = \{w_{2,n}\}$ be weights in $X$. Then the composition operator $C_\tau : l^{\phi_1}_{w_1}(\{a_n\}) \longrightarrow l^{\phi_2}_{w_2}(\{a_n\})$ is bounded if and only if there exist $a, b, \delta > 0$ and a sequence $\{c_n\}$ of nonnegative integers in $l^1$ such that $\phi_1(u)w_{1,n}a_n < \delta$ implies*

$$\phi_2(au)w_{2,\tau^{-1}(n)}a_nb_n \leq b\phi_1(u)w_{1,n}a_n + c_n$$

*for all $n \in \mathbb{N}$ and all $u \geq 0$.*

We divide the proof in two parts, the sufficient part and the necessary part. The proof of the necessary part is as under:

*Proof.* Suppose that the given condition holds. Let $\mathsf{f} = \{\mathsf{f}_n\}_{n=1}^{\infty} \in l^{\phi_1}_{w_1}(\{a_n\}) \setminus \{0\}$, then $I_{\phi_1,w_1}\left(\frac{\mathsf{f}}{\|\mathsf{f}\|_{\phi_1,w_1,\{a_n\}}}, \{a_n\}\right) \leq 1$. Let $M \geq 1$ be a real number satisfying $M(b + \|c\|_1) \geq 1$, where $\|c\|_1 = \sum\limits_{n=1}^{\infty} c_n$. Then

$$\begin{aligned}
&I_{\phi_2,w_2}\left(\frac{C_\tau \mathsf{f}}{(M(b+\|c\|_1)\|\mathsf{f}\|_{\phi_1,w_1,\{a_n\}})/a}, \{a_n\}\right) \\
&= \sum_{n=1}^{\infty} \phi_2\left(\frac{a|C_\tau \mathsf{f}_n|}{M(b+\|c\|_1)\|\mathsf{f}\|_{\phi_1,w_1,\{a_n\}}}\right) w_{2,n}a_n \\
&\leq \frac{1}{M(b+\|c\|_1)} \sum_{n=1}^{\infty} \phi_2\left(\frac{a|\mathsf{f}_{\tau(n)}|}{\|\mathsf{f}\|_{\phi_1,w_1,\{a_n\}}}\right) w_{2,n}a_n \\
&= \frac{1}{M(b+\|c\|_1)} \sum_{n\in\tau(X)} \phi_2\left(\frac{a|\mathsf{f}_n|}{\|\mathsf{f}\|_{\phi_1,w_1,\{a_n\}}}\right) w_{2,\tau^{-1}(n)}\mu(\tau^{-1}(A_n)) \\
&= \frac{1}{M(b+\|c\|_1)} \sum_{n=1}^{\infty} \phi_2\left(\frac{a|\mathsf{f}_n|}{\|\mathsf{f}\|_{\phi_1,w_1,\{a_n\}}}\right) w_{2,\tau^{-1}(n)}a_nb_n \\
&\leq \frac{1}{M(b+\|c\|_1)} \sum_{n=1}^{\infty} \left(b\ \phi_1\left(\frac{|\mathsf{f}_n|}{\|\mathsf{f}\|_{\phi_1,w_1,\{a_n\}}}\right) w_{1,n}a_n + c_n\right) \\
&\leq 1.
\end{aligned}$$

Thus $\|C_\tau \mathsf{f}\|_{\phi_2,\{a_n\}} \leq \frac{M}{a}(b + \|c\|_1)\|\mathsf{f}\|_{\phi_1,w_1,\{a_n\}}$. This shows that $C_\tau$ is bounded. □

**Definition 2.2.** Let us consider the sequences $\mathsf{z} = \{\mathsf{z}_n\}$, of real numbers. For $\alpha, \beta, \gamma, \delta > 0$, define

$$X^0_\alpha = \{\mathsf{z} \mid \text{there exists } k \in \mathbb{N} \text{ such that } \sum_{n=k}^{\infty} \phi_1(\alpha|\mathsf{z}_n|)w_{1,n}a_n < \infty\},$$

$$Y_\beta^0 = \{\mathsf{z} \mid \text{there exists } k \in \mathbb{N} \text{ such that } \sum_{n=k}^{\infty} \phi_2(\beta|\mathsf{z}_n|)w_{2,n}a_n < \infty\}$$

and $F_n(\alpha, \beta, \gamma, \delta)$

$$= \sup\{\phi_2(\beta|y|)w_{2,\tau^{-1}(n)}a_nb_n \mid \phi_1(\alpha|y|)w_{1,n}a_n < \min(\delta,\ \gamma^{-1}\phi_2(\beta|y|)w_{2,\tau^{-1}(n)}a_nb_n)\}.$$

If there is no $y$ for which $\phi_1(\alpha|y|)w_{1,n}a_n < \min(\delta,\ \gamma^{-1}\phi_2(\beta|y|)w_{2,\tau^{-1}(n)}a_nb_n)$, then we put $F_n(\alpha, \beta, \gamma, \delta) = 0$.

**Theorem 2.3.** *(i) If $0 < \alpha' < \alpha''$, then $X_{\alpha''}^0 \subset X_{\alpha'}^0$. Moreover, $\bigcap_\alpha X_\alpha^0 \neq \emptyset$.*

*(ii) $F_n(\alpha, \beta, \gamma, \delta)$ is nonincreasing with respect to $\alpha, \gamma$ and nondecreasing with respect to $\beta, \delta$.*

*(iii) If $\phi_1(\alpha|y|)w_{1,n}a_n < \delta$, then*

$$\phi_2(\beta|y|)w_{2,\tau^{-1}(n)}a_nb_n \leq \gamma\phi_1(\alpha|y|)w_{1,n}a_n + F_n(\alpha, \beta, \gamma, \delta).$$

*Proof.* (i) Let $\mathsf{z} = \{\mathsf{z}_n\}_{n=1}^{\infty} \in X_{\alpha''}^0$. Then there exists $k \in \mathbb{N}$ such that

$$\sum_{n=k}^{\infty} \phi_1(\alpha''|\mathsf{z}_n|)w_{1,n}a_n < \infty.$$

Since $\alpha' < \alpha''$, it follows that for each $n \in \mathbb{N}$,

$$\phi_1(\alpha'|\mathsf{z}_n|)w_{1,n}a_n \leq \phi_1(\alpha''|\mathsf{z}_n|)w_{1,n}a_n$$

which implies

$$\sum_{n=k}^{\infty} \phi_1(\alpha'|\mathsf{z}_n|)w_{1,n}a_n \leq \sum_{n=k}^{\infty} \phi_1(\alpha''|\mathsf{z}_n|)w_{1,n}a_n < \infty.$$

Thus $\mathsf{z} = \{\mathsf{z}_n\}_{n=1}^{\infty} \in X_{\alpha'}^0$.

(ii) Follows from the definition of $F_n(\alpha, \beta, \gamma, \delta)$.

(iii) Let us suppose that $\phi_1(\alpha|y|)w_{1,n}a_n < \delta$. If

$$\phi_1(\alpha|y|)w_{1,n}a_n < \min(\delta,\ \gamma^{-1}\phi_2(\beta|y|)w_{2,\tau^{-1}(n)}a_nb_n),$$

then $y \in E_n(\alpha, \beta, \gamma, \delta)$. Thus,

$$\phi_2(\beta|y|)w_{2,\tau^{-1}(n)}a_nb_n \leq F_n(\alpha, \beta, \gamma, \delta).$$

Since $\gamma\phi_1(\alpha|y|)w_{1,n}a_n \geq 0$, it follows that

$$\phi_2(\beta|y|)w_{2,\tau^{-1}(n)}a_nb_n \leq \gamma\phi_1(\alpha|y|)w_{1,n}a_n + F_n(\alpha, \beta, \gamma, \delta).$$

If

$$\phi_1(\alpha|y|)w_{1,n}a_n \not< \min(\delta,\ \gamma^{-1}\phi_2(\beta|y|)a_nb_n),$$

then

$$\gamma^{-1}\phi_2(\beta|y|)w_{2,\tau^{-1}(n)}a_nb_n \le \phi_1(\alpha|y|)w_{1,n}a_n,$$

which implies that

$$\phi_2(\beta|y|)w_{2,\tau^{-1}(n)}a_nb_n \le \gamma\phi_1(\alpha|y|)w_{1,n}a_n.$$

Again, since $F_n(\alpha,\beta,\gamma,\delta) \ge 0$, we obtain

$$\phi_2(\beta|y|)w_{2,\tau^{-1}(n)}a_nb_n \le \gamma\phi_1(\alpha|y|)w_{1,n}a_n + F_n(\alpha,\beta,\gamma,\delta).$$

□

**Theorem 2.4.** *Let $A$ and $B$ be two nonempty sets of positive numbers. If*

$$C_\tau\left(\bigcap_{\alpha\in A} X_\alpha^0\right) \subset \bigcup_{\beta\in B} Y_\beta^0,$$

*then there exist $\alpha\in A, \beta\in B, \gamma>0, \delta>0$ and $m\in\mathbb{N}$ such that*

$$\sum_{n=m}^{\infty} F_n(\alpha,\beta,\gamma,\delta) < \infty.$$

*Proof.* Suppose the theorem is not true. Choose $\alpha_k \in A$ and $\beta \in B$, $k=1,2,\ldots$, such that for all $\alpha\in A$ there is a $k'$ with $\alpha_k \ge \alpha$ for all $k>k'$, and for all $\beta\in B$ there is a $k''$ with $\beta_k \le \beta$ for all $k>k''$. Put

$$d(n,k) = F_n(\alpha_k,\beta_k,k,k^{-2}).$$

Then $\sum\limits_{n=m}^{\infty} d(n,k) = \infty$ for all $k$ and $m$. Let $n_1, n_2, \ldots$ be an increasing sequence of indices that partitions the set of natural numbers into segments $N_1 = \{1,2,\ldots,\ n_1\}$, $N_k = \{n_{k-1}+1,\ldots,n_k\}, k=2,3,\ldots$, such that

$$\sum_{m\in N_k} d(n,k) > \frac{1}{k} \tag{2.1}$$

$$\sum_{m\in N_k\setminus\{n_k\}} d(n,k) \le \frac{1}{k}, \tag{2.2}$$

for $k=1,2,\ldots$, where we put $\sum\limits_{n\in\emptyset} d(n,k) = 0$. Write

$$N_k^+ = \{n\in N_k \mid d(n,k)>0\}.$$

By the definition of $F_n$ and $d(n,k)$ and the inequality (2.1), for every $n\in N_k^+$, there exists $\mathsf{z}_n$ with the property that

$$\phi_1(\alpha_k|\mathsf{z}_n|)w_{1,n}a_n < \min(k^{-2},\ k^{-1}\phi_2(\beta_k|\mathsf{z}_n|)w_{2,\tau^{-1}(n)}a_nb_n)$$

such that

$$\sum_{n\in N_k^+} \phi_2(\beta|\mathsf{z}_n|)w_{2,\tau^{-1}(n)}a_nb_n > \frac{1}{k}. \tag{2.3}$$

By the definitions of $F_n$ and $d(n,k)$ and the inequality (2.2), we have

$$\begin{aligned} \sum_{n\in N_k^+} \phi_1(\alpha_k|\mathsf{z}_n|)w_{1,n}a_n &< \sum_{n\in N_k^+\setminus\{n_k\}} \frac{1}{k}\phi_2(\beta|\mathsf{z}_n|)w_{2,\tau^{-1}(n)}a_nb_n + \frac{1}{k^2} \\ &\leq \frac{1}{k}\sum_{n\in N_k^+\setminus\{n_k\}} d(n,k) + \frac{1}{k^2} \\ &\leq \frac{2}{k^2}. \end{aligned} \tag{2.4}$$

Thus, for each $n \in N_k^+$, there exists $\mathsf{z}_n \in \mathbb{R}$ such that (2.3) and (2.4) hold. For $n \in N_k \setminus N_k^+$, choose $\mathsf{z}_n \in \mathbb{R}$ such that $\sum_{n\in N_k\setminus N_k^+} \phi_1(\alpha|\mathsf{z}_n|)w_{1,n}a_n < \infty$. Let $\mathsf{z} = \{\mathsf{z}_n\}$. We now take an arbitrary $\alpha \in A$ and choose $k'$ so large that for all $k \geq k'$, we have $\alpha_k \geq k$. Then by using (2.4), we obtain

$$\begin{aligned} \sum_{n=n_{k'-1}+1}^{\infty} \phi_1(\alpha|\mathsf{z}_n|)w_{1,n}a_n &= \sum_{k=k'}^{\infty}\sum_{n\in N_k} \phi_1(\alpha|\mathsf{z}_n|)w_{1,n}a_n \\ &\leq \sum_{k=k'}^{\infty}\left[\sum_{n\in N_k} \phi_1(\alpha_k|\mathsf{z}_n|)w_{1,n}a_n + \sum_{n\in N_k\setminus N_k^+} \phi_1(\alpha|\mathsf{z}_n|)w_{1,n}a_n\right] \\ &< \infty. \end{aligned}$$

This shows that $\mathsf{z} \in \bigcap_{\alpha\in A} X_\alpha^0$. On the other hand, for any arbitrary $\beta \in B$ and $m \in \mathbb{N}$, choose $k''$ large enough that for all $k \geq k''$, we have $\beta \geq \beta_k$ and $n_{k-1}+1 \geq m$. By change of variable in the second step below and then readjusting the variable, and using (2.3) in the last step, we obtain

$$\begin{aligned} \sum_{n=m}^{\infty} \phi_2(\beta|C_\tau(\mathsf{z}_n)|)w_{2,n}a_n &= \sum_{n=m}^{\infty} \phi_2(\beta|\mathsf{z}_{\tau(n)}|)w_{2,n}a_n \\ &= \sum_{n=m}^{\infty} \phi_2(\beta|\mathsf{z}_n|)w_{2,\tau^{-1}(n)}\mu(\tau^{-1}(A_n)) \\ &= \sum_{n=m}^{\infty} \phi_2(\beta|\mathsf{z}_n|)w_{2,\tau^{-1}(n)}a_nb_n \\ &\geq \sum_{k=k'}^{\infty}\sum_{n\in N_k^+} \phi_2(\beta_k|\mathsf{z}_n|)w_{2,\tau^{-1}(n)}a_nb_n \\ &= \infty. \end{aligned}$$

Hence $C_\tau(\mathsf{z}) \notin \bigcup_{\beta\in B} Y_\beta^0$, which is a contradiction. □

**Corollary 2.5.** *(i) If $C_\tau(\bigcap\limits_{\alpha\in A} X_\alpha^0) \subset \bigcup\limits_{\beta\in B} Y_\beta^0$, then there exist $\alpha \in A, \beta \in B, \gamma > 0, \delta > 0$ and a sequence $\{c_n\}$ of nonnegative integers such that $\sum\limits_{n=m}^{\infty} c_n < \infty$ for some $m \in \mathbb{N}$ and that $\phi_1(\alpha|y|)w_{1,n}a_n < \delta$ implies*

$$\phi_2(\beta|y|)w_{2,\tau^{-1}(n)}a_nb_n \leq \gamma\phi_1(\alpha|y|)w_{1,n}a_n + c_n,$$

*for all $n \in \mathbb{N}$.*

*(ii) Suppose there exist $\alpha \in A, \beta \in B, \gamma > 0, \delta > 0$ and a sequence $\{c_n\}$ of nonnegative integers such that $\sum\limits_{n=m}^{\infty} c_n < \infty$ for some $m \in \mathbb{N}$ and that $\phi_1(\alpha|y|)w_{1,n}a_n < \delta$ implies*

$$\phi_2(\beta|y|)w_{2,\tau^{-1}(n)}a_nb_n \leq \gamma\phi_1(\alpha|y|)w_{1,n}a_n + c_n,$$

*for all $n \in \mathbb{N}$, then $C_\tau(X_\alpha^0) \subset Y_\beta^0$.*

*(iii) If $C_\tau(\bigcap\limits_{\alpha\in A} X_\alpha^0) \subset \bigcup\limits_{\beta\in B} Y_\beta^0$, then there exist $\alpha \in A$ and $\beta \in B$ such that $C_\tau(X_\alpha^0) \subset Y_\beta^0$.*

*(iv) $C_\tau(X_\alpha^0) \subset Y_\beta^0$ if and only if there exist $\gamma > 0, \delta > 0$ and a sequence $\{c_n\}$ of nonnegative integers such that $\sum\limits_{n=m}^{\infty} c_n < \infty$ for some $m \in \mathbb{N}$ and that $\phi_1(\alpha|y|)w_{1,n}a_n < \delta$ implies*

$$\phi_2(\beta|y|)w_{2,\tau^{-1}(n)}a_nb_n \leq \gamma\phi_1(\alpha|y|)w_{1,n}a_n + c_n,$$

*for all $n \in \mathbb{N}$.*

*(v) Let $X_\alpha = \{\mathsf{z} \mid \sum_{n=1}^{\infty} \phi_1(\alpha|\mathsf{z}_n|)w_{1,n}a_n < \infty\}$. If $C_\tau(X_\alpha) \subset \bigcup\limits_{\beta\in B} Y_\beta^0$, then there exists $\beta \in B$ such that $C_\tau(X_\alpha^0) \subset Y_\beta^0$. In particular, if $C_\tau(X_\alpha) \subset Y_\beta^0$, then $C_\tau(X_\alpha^0) \subset Y_\beta^0$.*

*Proof.* (i) Follows from Theorem 2.4 and Corollary 2.3(iii) with $F_n(\alpha, \beta, \gamma, \delta) = c_n$.

(ii) Let $\mathsf{z} = \{\mathsf{z}_n\}_{n=1}^{\infty} \in X_\alpha^0$. Then there exists $k \in \mathbb{N}$ such that

$$\sum_{n=k}^{\infty} \phi_1(\alpha|\mathsf{z}_n|)w_{1,n}a_n < \infty$$

where we suppose $k \geq m$ and $\phi_1(\alpha|\mathsf{z}_n|)w_{1,n}a_n\ \delta$ for $n \geq k$. Thus

$$\begin{aligned}
\sum_{n=k}^{\infty} \phi_2(\beta|C_\tau(\mathsf{z}_n)|)w_{2,n}a_n &= \sum_{n=k}^{\infty} \phi_2(\beta|\mathsf{z}_{\tau(n)}|)w_{2,n}a_n \\
&= \sum_{n=k}^{\infty} \phi_2(\beta|\mathsf{z}_n|)w_{2,\tau^{-1}(n)}\mu(\tau^{-1}(A_n)) \\
&= \sum_{n=k}^{\infty} \phi_2(\beta|\mathsf{z}_n|)w_{2,\tau^{-1}(n)}a_nb_n \\
&\leq \gamma\sum_{n=k}^{\infty} \phi_1(\alpha|\mathsf{z}_n|)w_{1,n}a_n + \sum_{n=k}^{\infty} c_n \\
&< \infty.
\end{aligned}$$

Thus, $C_\tau(\mathsf{z}) \in Y_\beta^0$ and hence $C_\tau(X_\alpha^0) \subset Y_\beta^0$.

$(iii)$ Follows from $(i)$ and Theorem 2.3(iii) with $F_n(\alpha,\beta,\gamma,\delta) = c_n$.

$(iv)$ The direct part follows from $(i)$ and the converse part from $(ii)$.

$(v)$ Suppose $C_\tau(X_\alpha) \subset \bigcup_{\beta\in B} Y_\beta^0$. Let $\mathsf{z} \in X_\alpha^0$. Then there exists $k \in \mathbb{N}$ such that

$$\sum_{n=k}^{\infty} \phi_1(\alpha|\mathsf{z}_n|)w_{1,n}a_n < \infty.$$

We put $\bar{\mathsf{z}}_n = \bar{y}_n$ for $n < k$ and $\bar{\mathsf{z}}_n = \mathsf{z}_n$ for $n \geq k$, where $\bar{y}_n$ is the sequence given by Definition 2.2. Then $\bar{\mathsf{z}} = \{\bar{\mathsf{z}}_n\}_{n=1}^\infty \in X_\alpha$. Thus, by our supposition $C_\tau(\bar{\mathsf{z}}) \in \bigcup_{\beta\in B} Y_\beta^0$, so there exists $\overline{\beta} \in B$ such that $C_\tau(\bar{\mathsf{z}}) \in Y_{\overline{\beta}}^0$. It then follows that $C_\tau(\mathsf{z}) \in Y_{\overline{\beta}}^0$ which implies that $C_\tau(\mathsf{z}) \in \bigcup_{\beta\in B} Y_\beta^0$. Therefore $C_\tau(X_\alpha^0) \subset \bigcup_{\beta\in B} Y_\beta^0$. Finally, using $(iii)$, there exists $\beta \in B$ such that $C_\tau(X_\alpha^0) \subset Y_\beta^0$.

□

We now prove the sufficient part of Theorem 2.1.

**Theorem 2.6.** *If a composition operator $C_\tau : l_{w_1}^{\phi_1}(\{a_n\}) \longrightarrow l_{w_2}^{\phi_2}(\{a_n\})$ is bounded, then there exist $a, b, \delta > 0$ and a sequence $\{c_n\}_{n=1}^\infty$ of nonnegative integers in $l^1$ such that $\phi_1(u)w_{1,n}a_n < \delta$ implies*

$$\phi_2(au)w_{2,\tau^{-1}(n)}a_nb_n \leq b\phi_1(u)w_{1,n}a_n + c_n,$$

*for all $n \in \mathbb{N}$ and all $u \geq 0$.*

*Proof.* Observe that $\bigcup_{\alpha>0} X_\alpha^0 = l^{\phi_1}(\{a_n\})$ and $\bigcup_{\beta>0} Y_\beta^0 = l^{\phi_2}(\{a_n\})$. Suppose $C_\tau : l^{\phi_1}(\{a_n\}) \longrightarrow l^{\phi_2}(\{a_n\})$ is bounded. Then, by Corollary 2.5, we see that for each $\alpha > 0$, there exist $\beta, \gamma, \delta > 0$ and a sequence $\{c_n\}$ of nonnegative integers such that $\sum_{n=m}^{\infty} c_n < \infty$ for some $m \in \mathbb{N}$ and that $\phi_1(\alpha u)w_{1,n}a_n < \delta$ implies

$$\phi_2(\beta u)w_{2,\tau^{-1}(n)}a_nb_n \leq \gamma\phi_1(\alpha u)w_{1,n}a_n + c_n,$$

for all $n \in \mathbb{N}$ and all $u \geq 0$.
Putting $\alpha u = v, \beta/\alpha = a$ and $\gamma = b$, the above condition can be rewritten as $\phi_1(v)w_{1,n}a_n < \delta$ implies

$$\phi_2(av)w_{2,\tau^{-1}(n)}a_nb_n \leq b\phi_1(v)w_{1,n}a_n + c_n,$$

for all $n \in \mathbb{N}$ and all $v \geq 0$.
For $n < m$, we take $c_n = \max[\sup_{v\in S}(\phi_2(av)w_{2,\tau^{-1}(n)}a_nb_n - b\phi_1(v)w_{1,n}a_n),\ 0]$, where $S$ is the compact set of all $v \geq 0$ such that $\phi_1(v)w_{1,n}a_n < \delta$. Thus, $\{c_n\}_{n=1}^\infty \in l^1$, which yields the desired result. □

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