On Certain Integrals Involving Spheroidal, Mathieu and Multivariable Gimel-Functions

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ABSTRACT
The aim of this note is to evaluate certain interesting new integrals involving spheroidal, Mathieu and multivariable Gimel-functions. The importance of the results established in this paper lies in the usefulness of spheroidal and Mathieu functions in the multiplicity of Physical problems as also of the most general nature of the multivariable Gimel-function.

KEYWORDS : Multivariable Gimel-function, multiple integral contours, spheroidal function, Mathieu function.

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1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C}, \mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The solution of the following differential equations

$$(x^2-1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + (c^2x^2 - \lambda_{0,n}(c))y = 0 \tag{1.1}$$

and

$$\frac{d^2y}{dx^2} - (a - 2q\cosh 2x)y = 0 \tag{1.2}$$

are respectively called radial prolate spheroidal functions of order zero and modified Mathieu functions. The whole theory, asymptotic expansions and special properties of these functions which make them extra-ordinarily useful in certain branches of Physics and Mechanics has been given in an excellent and coordinated manner in the standard works of Flammer [4] and McLachlan [5].

Prolate spheroidal function of order zero has been defined and represented by Flammer [4] in the following way :

$$R_{0n}^{(1)}(c,x) = \frac{1}{\sum_{u=0,1}^{\infty'} d_u^{0n}(c)} \sum_{u=0,1}^{\infty'} d_u^{0n}(c)\omega^{u-n}\sqrt{\frac{\pi}{2cx}}J_{u+\frac{1}{2}}(cx) \tag{1.3}$$

wher prime over the summation indicates that the summation is over only even of u when n is even, and over only odd values of u when n is odd. Also d_u^{0n} are the solutions of second order difference differential equations [4].
Following McLachlan [5], we shall define and represent modified Mathieu functions which are solutions of (1.2) in the following manner :

$$Ce_{2n}(x,q) = \frac{ce_{2n}(0,q)}{A_0^{(2n)}} \sum_{u=0}^{\infty} A_{2u}^{(2n)} J_{2u}(2k.\sinh x) \tag{1.4}$$

The form of the coefficients $A_{2u}^{(2n)}$ for various values of u and n and small values of q can be referred to in Mclachlan's book [5].

We define a generalized transcendental function of several complex variables noted $\gimel$.

$$\daleth(z_1,\cdots,z_r)=\daleth^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|}z_1\\ \cdot\\ \cdot\\ \cdot\\ z_r\end{array}\right.$$

$$[(a_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}};[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$
$$[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}};$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(a_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;\qquad\cdots$$

$$[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{n_r+1,p_r}]:\ [(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}]$$
$$[\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{1,q_r}]:\ [(d_j^{(1)}),\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}]$$

$$;\cdots;[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}]$$
$$\left.;\cdots;[(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,n^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{n^{(r)}+1,q_i^{(r)}}]\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.5}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha_{2ji_2}^{(k)}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta_{2ji2}^{(k)}s_k)]}$$

$$\frac{\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.6}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.7}$$

1) $[(c_j^{(1)};\gamma_j^{(1)}]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $n_2,\cdots,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2,\cdots,0\leqslant m_r,0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$,

$0\leqslant n^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant p_{i^{(r)}}$

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r)$;

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n^{(k)});(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$a_{kji_k}\in\mathbb{C};(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r)$.

$b_{kji_k}\in\mathbb{C};(j=1,\cdots,q_{i_k});(k=2,\cdots,r)$.

$d_{ji^{(k)}}^{(k)}\in\mathbb{C};(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{C};(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

The contour L_k is in the $s_k(k=1,\cdots,r)$- plane and run from $\sigma-i\infty$ to $\sigma+i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k\right)(j=1,\cdots,n_2),\Gamma^{A_{3j}}\left(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k\right)$ $(j=1,\cdots,n_3),\cdots,\Gamma^{A_{rj}}\left(1-a_{rj}+\sum_{i=1}^{r}\alpha_{rj}^{(i)}\right)(j=1,\cdots,n_r),\Gamma^{C_j^{(k)}}\left(1-c_j^{(k)}+\gamma_j^{(k)}s_k\right)(j=1,\cdots,n^{(k)})(k=1,\cdots,r)$to the right of the contour L_k and the poles of $\Gamma^{D_j^{(k)}}\left(d_j^{(k)}-\delta_j^{(k)}s_k\right)(j=1,\cdots,m^{(k)})(k=1,\cdots,r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)|<\frac{1}{2}A_i^{(k)}\pi$ where

$$A_i^{(k)}=\sum_{j=1}^{m^{(k)}}D_j^{(k)}\delta_j^{(k)}+\sum_{j=1}^{n^{(k)}}C_j^{(k)}\gamma_j^{(k)}-\tau_{i^{(k)}}\left(\sum_{j=m^{(k)}+1}^{q_i^{(k)}}D_{ji^{(k)}}^{(k)}\delta_{ji^{(k)}}^{(k)}+\sum_{j=n^{(k)}+1}^{p_i^{(k)}}C_{ji^{(k)}}^{(k)}\gamma_{ji^{(k)}}^{(k)}\right)+$$

$$-\tau_{i_2}\left(\sum_{j=n_2+1}^{p_{i_2}}A_{2ji_2}\alpha_{2ji_2}^{(k)}+\sum_{j=1}^{q_{i_2}}B_{2ji_2}\beta_{2ji_2}^{(k)}\right)-\cdots-\tau_{i_r}\left(\sum_{j=n_r+1}^{p_{i_r}}A_{rji_r}\alpha_{rji_r}^{(k)}+\sum_{j=1}^{q_{i_r}}B_{rji_r}\beta_{rji_r}^{(k)}\right) \tag{1.8}$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,),max(\,|z_1|,\cdots,|z_r|\,)\rightarrow 0$$

$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,),min(\,|z_1|,\cdots,|z_r|\,)\rightarrow\infty$ where $i=1,\cdots,r:$

$\alpha_i=\min\limits_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\dfrac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]$ and $\beta_i=\max\limits_{1\leqslant j\leqslant n^{(i)}}Re\left[C_j^{(i)}\left(\dfrac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)\right]$

Remark 1.
If $n_2=\cdots=n_{r-1}=p_{i_2}=q_{i_2}=\cdots=p_{i_{r-1}}=q_{i_{r-1}}=0$ and $A_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=A_{rji_r}=B_{rji_r}=1$ $A_{rj}=A_{rji_r}=B_{rji_r}=1$, then the multivariable Gimel-function reduces in the multivariable Aleph- function defined by Ayant [1].

Remark 2.
If $n_2=\cdots=n_r=p_{i_2}=q_{i_2}=\cdots=p_{i_r}=q_{i_r}=0$ and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2=\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [7].

Remark 3.
If$A_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=A_{rji_r}=B_{rji_r}=1$and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2=\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [6].

Remark 4.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H-function defined by Srivastava and panda [8,9].

In your investigation, we shall use the following notations.

$$\mathbb{A}=[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.9}$$

$$\mathbf{A}=[(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.10}$$

$$A=[(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \tag{1.11}$$

$$\mathbb{B}=[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta^{(1)}_{(r-1)ji_{r-1}},\cdots,\beta^{(r-1)}_{(r-1)ji_{r-1}};B_{(r-1)ji_{r-1}})_{1,q_{i_{r-1}}}] \tag{1.12}$$

$$\mathbf{B}= \ [\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{1,q_{i_r}}] \tag{1.13}$$

$$\mathrm{B}=[(\mathrm{d}^{(1)}_j,\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}];\cdots;$$

$$[(\mathrm{d}^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})_{m^{(r)}+1,q^{(r)}_i}] \tag{1.14}$$

$$U=0,n_2;0,n_3;\cdots;0,n_{r-1};V=m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.15}$$

$$X=p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.16}$$

2. Main integrals.

In this section, we evaluate two integrals.

Theorem 1.

$$\int_0^\infty x^{\sigma-1}\gimel(a_1(cx)^{h_1},\cdots,a_r(cx)^{h_r})R^1_{0n}(c,x)dx=c^{-\sigma}\sqrt{\pi}2^{\sigma-2}\frac{1}{\sum_{u=0,1}^{\infty'}d^{0n}_u(c)}\,\omega^{u-n}\sum_{u=0,1}^{\infty'}d^{0n}_u(c)$$

$$\gimel^{U;0,n_r+2:V}_{X;p_{i_r}+1,q_{i_r},\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} 2^{h_1}a_1 & \mathbb{A};\left(\frac{2-\sigma-u}{2};\frac{h_1}{2},\cdots,\frac{h_r}{2};1\right),\mathbf{A},\left(\frac{u-\sigma+3}{2};\frac{h_1}{2},\cdots,\frac{h_r}{2};1\right):A \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{h_r}a_r & \mathbb{B};\mathbf{B}:\mathrm{B} \end{array}\right) \tag{2.1}$$

Provided

$c>0,h_i>0(i=1,\cdots,r),-u<Re(\sigma)<2.\ Re(\sigma+u)+\sum_{i=1}^r h_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D^{(i)}_j\left(\frac{d^{(i)}_j}{\delta^{(i)}_j}\right)\right]>0$

$Re(\sigma-2)+\sum_{i=1}^r h_i\max_{1\leqslant j\leqslant n^{(i)}}Re\left[C^{(i)}_j\left(\frac{c^{(i)}_j-1}{\gamma^{(i)}_j}\right)\right]<0,\ |arg(a_i(cx)^{h_i})|<\frac{1}{2}A^{(k)}_i\pi$ where $A^{(k)}_i$ is defined by (1.8) and the series on the right-hand side of (2.1) is absolutely convergent.

Proof

Multiply the left hand side of (2.1) by c^σ and substitute the value of R^1_{0n} in it from (1.3) to obtain (say J)

$$J=\int_0^\infty (cx)^{\sigma-1}\gimel(a_1(cx)^{h_1},\cdots,a_r(cx)^{h_r})\times\left[\frac{1}{\sum_{u=0,1}^{\infty'}d^{0n}_u(c)}\sum_{u=0,1}^{\infty'}d^{0n}_u(c)\omega^{u-n}\sqrt{\frac{\pi}{2cx}}J_{u+\frac{1}{2}}(cx)\right]\mathrm{cd}x \tag{2.2}$$

On replacing cx by z in (2.2) and interchanging the order of integration and summation therein we get

$$J=\frac{\sqrt{\frac{\pi}{2}}}{\sum_{u=0,1}^{\infty'}d^{0n}_u(c)}\sum_{u=0,1}^{\infty'}d^{0n}_u(c)\omega^{u-n}\sqrt{\frac{\pi}{2cx}}\int_0^\infty z^{\sigma-\frac{3}{2}}\gimel(a_1z^{h_1},\cdots,a_rz^{h_r})J_{u+\frac{1}{2}}(z)\mathrm{d}z \tag{2.3}$$

Now expressing the multivariable Gimel-function occurring in (2.3) in terms of Mellin-Barnes multiple integrals contour with the help of (1.5), interchanging the order of integrations (which is permissible under the conditions mentioned in (2.1)), we get

$$J = \frac{\sqrt{\frac{\pi}{2}}}{\sum_{u=0,1}^{\infty'} d_u^{0n}(c)} \sum_{u=0,1}^{\infty'} d_u^{0n}(c)\omega^{u-n}\sqrt{\frac{\pi}{2cx}} \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1,\cdots,s_r) \prod_{i=1}^r \theta_i(s_i) a_i^{s_i}$$

$$\left[\int_0^\infty z^{\sigma+\sum_{i=1}^r h_i s_i - \frac{3}{2}} J_{u+\frac{1}{2}}(z)\mathrm{d}z\right]\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.4}$$

On evaluating the z-integral involved in the above equation with the help of the result [3] and interpreting the Mellin-Barnes multiple integrals contour in terms of the multivariable Gimel-function, we get the desired result (2.1).
All that remains now is to justify the interchange of order of integration and summation in (2.2). This is possible, since

1) the series expansion of $R_{0,n}^1$ viz $\frac{1}{\sum_{u=0,1}^{\infty'} d_u^{0n}(c)} \sum_{u=0,1}^{\infty'} d_u^{0n}(c)\omega^{u-n}\sqrt{\frac{\pi}{2cx}} J_{u+\frac{1}{2}}(cx)$ is uniformly convergent [4] in any fixed interval $0 \leqslant x \leqslant b$.

2) the function $x^{\sigma-1}\gimel(a_1(cx)^{h_1},\cdots,a_r(cx)^{h_r})$ is continuous

3) The integral on the right hand side of (2.2) is absolutely convergent under the existence conditions mentioned with (2.1).

Theorem 2.

$$\int_0^\infty (\sinh x)^{\sigma-1}\gimel(a_1(2k\sinh x)^{h_1},\cdots,a_r(2k\sinh x)^{h_r}) Ce_{2n}(x,q)\cosh x\mathrm{d}x = \frac{\mathrm{k}^{-\sigma}\mathrm{ce}_{2\mathrm{n}}(0,\mathrm{q})}{2\mathrm{A}_0^{(2\mathrm{n})}} \sum_{u=0}^\infty A_{2u}^{(2n)}$$

$$\gimel_{X;p_{i_r}+1,q_{i_r},\tau_{i_r}:R_r:Y}^{U;0,n_r+2:V}\left(\begin{array}{c|c} 2^{h_1}a_1 & \mathbb{A};\ \left(\frac{2-\sigma-2u}{2};\frac{h_1}{2},\cdots,\frac{h_r}{2};1\right),\mathbf{A},\left(\frac{2u-\sigma+2}{2};\frac{h_1}{2},\cdots,\frac{h_r}{2};1\right):A \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{h_r}a_r & \mathbb{B};\mathbf{B}:\mathrm{B} \end{array}\right) \tag{2.5}$$

Provided

$$h>0, h_i>0 (i=1,\cdots,r), -2u<Re(\sigma)<2.\ Re(\sigma+2u)+\sum_{i=1}^r h_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$$

$Re\left(\sigma-\frac{3}{2}\right)+\sum_{i=1}^r h_i \max_{1\leqslant j\leqslant n^{(i)}} Re\left[C_j^{(i)}\left(\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)\right]<0,\ |arg(a_i(2k\sinh x)^{h_i})|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.8) and the series on the right-hand side of (2.5) is absolutely convergent.

The theorem 2 can be evaluated by proceeding in a similar manner to that given above. The series expansion for modified Mathieu function is also uniformly convergent [5].

3. Special cases.

If in (2.1) we let $c\to 0, x\to\infty$ in such a way that cx remains finite say z we have [4]

$$d_u^{0n}(0) = \left[\begin{array}{l} 0 \text{ if u} \neq n \\ \quad\quad . \\ 1 \text{ if u} = \mathrm{n} \end{array}\right. \tag{3.1}$$

$R_{0,n}^1$ occurring in (2.1) degenerates into $\sqrt{\frac{\pi}{2z}}J_{n+\frac{1}{2}}(z)$ and we have the following result

Corollary.

$$\int_0^{\infty} z^{\sigma-\frac{3}{2}}\gimel(a_1 z^{h_1},\cdots,a_r z^{h_r})J_{n+\frac{1}{2}}dz = 2^{\sigma-\frac{3}{2}}$$

$$\gimel^{U;0,n_r+2:V}_{X;p_{i_r}+1,q_{i_r},\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} 2^{h_1}a_1 & \mathbb{A};\left(\frac{2-\sigma-n}{2};\frac{h_1}{2},\cdots,\frac{h_r}{2};1\right),\mathbf{A},\left(\frac{n-\sigma+3}{2};\frac{h_1}{2},\cdots,\frac{h_r}{2};1\right):A \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{h_r}a_r & \mathbb{B};\mathbf{B}:B \end{array}\right) \tag{3.1}$$

Provided

$$h>0, h_i>0 (i=1,\cdots,r), -n<Re(\sigma)<2.\ Re(\sigma+n)+\sum_{i=1}^{r}h_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$$

$Re(\sigma-2)+\sum_{i=1}^{r}h_i \max_{1\leqslant j\leqslant n^{(i)}} Re\left[C_j^{(i)}\left(\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)\right]<0,\ |arg(a_i(cx)^{h_i})|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.8) and the series on the right-hand side of (2.1) is absolutely convergent.

Again if $k\rightarrow 0, x\rightarrow\infty$ in (2.5) in such a way that $2k\sinh x$ is finite and tends to y, $Ce_{2n}(x,q)\rightarrow J_{2n}(y)$ and the integral (2.5) yields a similar integral to the above integral (3.1) but having the integral order of Bessel function in it.

4. Conclusion.

A large number of other integrals involving spheroidal and modified Mathieu functions can also be obtained from (2.1) and (2.5) on account of the most general nature of the multivariable Gimel-function. This function can be suitably specialized to a remarkably wide variety of special functions (or product of several such special functions) which are expressible in terms of E, F, G, H, I, Aleph-functions of one or more variables.

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