

On Transformation Formulae for Basic Hypergeometric Functions by using WP Baley's Pairs

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Abstract:- In this paper making use of well known WP Bailey's Pairs, an attempt has been made to establish certain interesting transformation formulae for Basic Hypergeometric functions.

Key words:- *Bailey's transform, Bailey Pairs, WP Bailey Pairs, transformations formulae.*

1. Introduction

Transformation theory play very important role in the theory of q-hypergeometric series. Rogers-Ramanujan type identities are established through transformation formulae and identities have great importance in the theory of partitions.

By using WP Bailey's pairs S.N. Singh [1] established transformation formulae for q-series, as

$$(1.1) \quad {}_4\Phi_3 \left[\begin{matrix} k, ky/a, kz/a, aq/yz; q; a^2 q/k^2 \\ aq/y, aq/z, kyz/a \end{matrix} \right] = \frac{(aq/k, a^2 q/k; q)_\infty}{(aq, a^2 q/k^2; q)_\infty} \\ \times {}_{10}\Phi_9 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, y, z, a^2 q/kyz, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}; q; aq/k \\ \sqrt{a}, -\sqrt{a}, qa/k, qz/k, qy/k, qz/a, qy/a, qz/\sqrt{k}, qy/\sqrt{k}, qz/a\sqrt{k}, qy/a\sqrt{k} \end{matrix} \right]$$

$$(1.2) \quad \frac{(aq, a^2q/k^2; q)_\infty}{(aq/k, a^2q/k; q)_\infty} \times \left(\frac{k+a}{k}\right) {}_4\Phi_3 \left[\begin{matrix} k, ky/a, kz/a, aq/yz; q; a^2/k \\ aq/y, aq/z, kyz/a \end{matrix} \right]$$

$$= 10\Phi_9 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, y, z, a^2 q / k\rho_1\rho_2, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}; q; a/k \\ \sqrt{a}, -\sqrt{a}, aq / y, aq / z, k\rho_1\rho_2 / a, aq / \sqrt{k}, -aq / \sqrt{k}, a / \sqrt{k / a}, -a / \sqrt{k / a} \end{matrix} \right]$$

Here an attempt has been made to establish certain interesting transformation formulae for Basic Hypergeometric functions by using some WP Bailey's pairs.

2. Notation

Consider $|q| < 1$, where q is non-zero complex number, this condition ensures all the infinite products that we use will converge. We will use the notation,

$$(2.1) \quad (\alpha; q)_n = \begin{cases} (1 - \alpha)(1 - \alpha q) \dots \dots \dots (1 - \alpha q^{n-1}); & n > 0 \\ 1; & n = 0, \end{cases}$$

$$(2.2) \quad (\alpha; q)_\infty \equiv \prod_{r=0}^{\infty} (1 - \alpha q^r).$$

$$(2.3) \quad (\alpha; q)_{-n} = \frac{(-)^n q^{n(n+1)/2}}{\alpha^n [q/q : q]_n},$$

$$(2.4) \quad (z; q)_{2n} \equiv (z; q)_n (zq^n; q)_n$$

$$(2.5) \quad (z_1, z_2, \dots, z_n; q)_\infty \equiv (z_1; q)_\infty (z_2; q)_\infty \cdots (z_n; q)_\infty$$

Following the above notation, we define

$$(2.6) \quad {}_r\phi_s \left[\begin{matrix} a_1, a_2, a_3, \dots, a_r; q \\ b_1, b_2, \dots, b_s \end{matrix}; z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, a_3, \dots, a_r; q)_n Z^n}{(q, b_1, b_2, \dots, b_s; q)_n}$$

Max(|q|, |z| < 1). where

$$(a_1, a_2, \dots, a_r; q)_n = (a_1; q)_n (a_2; q)_n \dots (a_r; q)_n$$

A basic series in which the product of each pair of numerator and denominator parameters is constant, is called a well poised basic series. For example,

$${}_3\phi_2 \left[\begin{matrix} a, b, c \\ aq/b, aq/c \end{matrix}; z \right], \quad \text{is said to be a series well-poised in } aq.$$

W.N.Bailey in 1944 stated a theorem which is simple but very use full. If

$$(2.7) \quad \beta_n = \sum_{r=0}^n \alpha_r u_{n-r} v_{n+r}$$

$$(2.8) \quad \gamma_n = \sum_{r=0}^{\infty} \delta_{r+n} u_n v_{r+2n}$$

where $\alpha_r, \beta_r, u_r, \text{and } v_r$ are functions of r alone , such that the series γ_n exists, then

$$(2.9) \quad \sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n$$

We shall make use of the following known WP Bailey's Pairs

$$(2.10) \quad \alpha_n(a, k; q) = \frac{(a, q\sqrt{a}, -q\sqrt{a}, a/k, \rho_1, \rho_2; q)_n}{(q, \sqrt{a}, -\sqrt{a}, kq, aq/\rho_1, aq/\rho_2; q)_n} \left(\frac{k}{a}\right)^n,$$

$$\& \beta_n(a, k; q) = \frac{[k, k/m, mq/\rho_1, mq/\rho_2; q]_n}{[q, mq, aq/\rho_1, aq/\rho_2; q]_n},$$

where $m=k\rho_1\rho_2/aq$
[Mishra B. P.;6]

$$(2.11) \quad \alpha_n(a, k; q) = \frac{(a, q\sqrt{a}, -q\sqrt{a}, b, c, a^2q/bcm, \rho_1, \rho_2; q)_n}{(q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bcm/a, aq/\rho_1, aq/\rho_2; q)_n} \left(\frac{k}{a}\right)^n,$$

$$\& \beta_n(a, k; q) = \frac{(k, k/m, mq/\rho_1, mq/\rho_2; q)_n}{(q, mq, aq/\rho_1, aq/\rho_2; q)_n} \times \sum_{r=0}^n \left(\frac{1-mq^{2r}}{1-m}\right) \frac{(m, mb/a, mc/a, aq/bc, \rho_1, \rho_2, kq^n, q^{-n}; q)_r q^r}{(q, aq/b, aq/c, mbc/a, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{1+n}; q)_r},$$

where $m=k\rho_1\rho_2/aq$
[Mishra B. P.;6]

We shall also make use of the following well known theorems.

(2.12) Theorem 1. If $\{\alpha_n(a, k; q), \beta_n(a, k; q)\}$ is a W.P. Bailey pair then

$$\sum_{n=0}^{\infty} \left(\frac{a^2q}{k^2}\right)^n \beta_n(a, k; q) = \frac{(aq/k, a^2q/k; q)_{\infty}}{(aq, a^2q/k^2; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(k; q)_{2n}}{(a^2q/k; q)_{2n}} \left(\frac{a^2q}{k^2}\right)^n \alpha_n(a, k; q)$$

[Lauglin;4(1.7)]

(2.13) Theorem 2. If $\{\alpha_n(a, k; q), \beta_n(a, k; q)\}$ is a W.P. Bailey pair then

$$\sum_{n=0}^{\infty} \left(\frac{a^2}{k^2}\right)^n \beta_n(a, k; q) = \frac{(aq/k, a^2q/k; q)_{\infty}}{(aq, a^2q/k^2; q)_{\infty}} \left(\frac{k}{k+a}\right) \sum_{n=0}^{\infty} \frac{(k; q)_{2n}}{(a^2q/k; q)_{2n}} (1 + aq^{2n}) \left(\frac{a^2}{k^2}\right)^n \alpha_n(a, k; q)$$

[Singh, S.N.7;theorem 3]

(2.14) Theorem 3. If $\{\alpha_n(a, k; q), \beta_n(a, k; q)\}$ is a W.P. Bailey pair then

$$\sum_{n=0}^{\infty} \left(\frac{1-kq^{2n}}{1-k} \right) \frac{(\rho_1, \rho_2; q)_n}{(kq/\rho_1, kq/\rho_2; q)_n} \left(\frac{aq}{\rho_1 \rho_2} \right)^n \beta_n(a, k; q)$$

$$= \frac{(kq, kq/\rho_1 \rho_2, aq/\rho_1, aq/\rho_2; q)_\infty}{(aq, aq/\rho_1 \rho_2, kq/\rho_1, kq/\rho_2; q)_\infty} \sum_{n=0}^{\infty} \frac{(\rho_1, \rho_2; q)_n}{(aq/\rho_1, aq/\rho_2; q)_n} \left(\frac{aq}{\rho_1 \rho_2} \right)^n \alpha_n(a, k; q)$$

[Lauglin 5; theorem 1]

3. Main Results

$$(3.1) {}_4\Phi_3 \left[\begin{matrix} k, k/m, mq/\rho_1, mq/\rho_2; q; a^2 q/k^2 \\ mq, aq/\rho_1, aq/\rho_2 \end{matrix} \right] = \frac{[aq/k, a^2 q/k; q]_\infty}{[aq, a^2 q/k^2; q]_\infty} \sum_{n=0}^{\infty} \frac{(a, q\sqrt{a}, -q\sqrt{a}, a/k, \rho_1, \rho_2; q)_n (k, kq; q^2)_n}{(q, \sqrt{a}, -\sqrt{a}, kq, aq/\rho_1, aq/\rho_2; q)_n (a^2 q/k, a^2 q^2/k; q^2)_n} \left(\frac{aq}{k} \right)^n$$

where $m = k\rho_1 \rho_2 / aq$

$$(3.2) {}_4\Phi_3 \left[\begin{matrix} k, k/m, mq/\rho_1, mq/\rho_2; q; a^2 q/k^2 \\ mq, aq/\rho_1, aq/\rho_2 \end{matrix} \right]$$

$$= \frac{(aq/k, a^2 q/k; q)_\infty}{(aq, a^2 q/k^2; q)_\infty} \left(\frac{k}{k+a} \right) \sum_{n=0}^{\infty} \frac{(a, q\sqrt{a}, -q\sqrt{a}, a/k, \rho_1, \rho_2; q)_n (k, kq; q^2)_n}{(q, \sqrt{a}, -\sqrt{a}, kq, aq/\rho_1, aq/\rho_2; q)_n (a^2 q/k, a^2 q^2/k; q^2)_n} \left(\frac{aq}{k} \right)^n$$

$$+ \frac{(aq/k, a^2 q/k; q)_\infty}{(aq, a^2 q/k^2; q)_\infty} \left(\frac{ak}{k+a} \right) \sum_{n=0}^{\infty} \frac{(a, q\sqrt{a}, -q\sqrt{a}, a/k, \rho_1, \rho_2; q)_n (k, kq; q^2)_n}{(q, \sqrt{a}, -\sqrt{a}, kq, aq/\rho_1, aq/\rho_2; q)_n (a^2 q/k, a^2 q^2/k; q^2)_n} \left(\frac{aq^2}{k} \right)^n$$

where $m = k\rho_1 \rho_2 / aq$

$$(3.3) {}_6\Phi_5 \left[\begin{matrix} k, k/m, mq/\rho_1, mq/\rho_2, \rho_1, \rho_2; q; aq/\rho_1 \rho_2 \\ mq, kq/\rho_1, kq/\rho_2, aq/\rho_1, aq/\rho_2, \end{matrix} \right] - k {}_6\Phi_5 \left[\begin{matrix} k, k/m, mq/\rho_1, mq/\rho_2, \rho_1, \rho_2; q; aq^3/\rho_1 \rho_2 \\ mq, kq/\rho_1, kq/\rho_2, aq/\rho_1, aq/\rho_2, \end{matrix} \right]$$

$$= \frac{[k, kq/\rho_1 \rho_2, aq/\rho_1, aq/\rho_2; q]_\infty}{[aq, aq/\rho_1 \rho_2, kq/\rho_1, kq/\rho_2; q]_\infty} {}_8\Phi_7 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, a/k, \rho_1, \rho_2, \rho_1, \rho_2; q; kq/\rho_1 \rho_2 \\ \sqrt{a}, -\sqrt{a}, kq, aq/\rho_1, aq/\rho_2, aq/\rho_1, aq/\rho_2 \end{matrix} \right]$$

where $m = k\rho_1 \rho_2 / aq$

$$(3.4) \sum_{n=0}^{\infty} \frac{(k, k/m, mq/\rho_1, mq/\rho_2; q)_n}{(q, mq, aq/\rho_1, aq/\rho_2; q)_n} \left(\frac{a^2 q}{k^2} \right)^n {}_8\Phi_7 \left[\begin{matrix} m, aq/bc, mb/a, mc/a, \rho_1, \rho_2, kq^n, q^{-n}; q; q \\ aq/b, aq/c, mbc/a, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{1+n} \end{matrix} \right]_n$$

$$- m \sum_{n=0}^{\infty} \frac{(k, k/m, mq/\rho_1, mq/\rho_2; q)_n}{(q, mq, aq/\rho_1, aq/\rho_2; q)_n} \left(\frac{a^2 q}{k^2} \right)^n {}_8\Phi_7 \left[\begin{matrix} m, aq/bc, mb/a, mc/a, \rho_1, \rho_2, kq^n, q^{-n}; q; q^3 \\ aq/b, aq/c, mbc/a, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{1+n} \end{matrix} \right]_n$$

$$= \frac{(m, aq/k, a^2 q/k; q)_\infty}{(mq, aq, a^2 q/k^2; q)_\infty} \sum_{n=0}^{\infty} \frac{(a, q\sqrt{a}, -q\sqrt{a}, b, c, a^2 q/bcm, \rho_1, \rho_2; q)_n (k, kq; q^2)_n}{(q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bcm/a, aq/\rho_1, aq/\rho_2; q)_n (a^2 q/k, a^2 q^2/k; q^2)_n} \left(\frac{aq}{k} \right)^n$$

where $m = k\rho_1 \rho_2 / aq$

$$(3.5) \sum_{n=0}^{\infty} \frac{(k, k/m, mq/\rho_1, mq/\rho_2; q)_n}{(q, mq, aq/\rho_1, aq/\rho_2; q)_n} \left(\frac{a^2}{k^2} \right)^n {}_8\Phi_7 \left[\begin{matrix} m, aq/bc, mb/a, mc/a, \rho_1, \rho_2, kq^n, q^{-n}; q; q \\ aq/b, aq/c, mbc/a, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{1+n} \end{matrix} \right]_n$$

$$- m \sum_{n=0}^{\infty} \frac{(k, k/m, mq/\rho_1, mq/\rho_2; q)_n}{(q, mq, aq/\rho_1, aq/\rho_2; q)_n} \left(\frac{a^2}{k^2} \right)^n {}_8\Phi_7 \left[\begin{matrix} m, aq/bc, mb/a, mc/a, \rho_1, \rho_2, kq^n, q^{-n}; q; q^3 \\ aq/b, aq/c, mbc/a, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{1+n} \end{matrix} \right]_n$$

$$= \frac{(m, aq/k, a^2 q/k; q)_\infty}{(mq, aq, a^2 q/k^2; q)_\infty} \left(\frac{ak}{k+a} \right) \sum_{n=0}^{\infty} \frac{(a, q\sqrt{a}, -q\sqrt{a}, b, c, a^2 q/bcm, \rho_1, \rho_2; q)_n (k, kq; q^2)_n}{(q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bcm/a, aq/\rho_1, aq/\rho_2; q)_n (a^2 q/k, a^2 q^2/k; q^2)_n} \left(\frac{a}{k} \right)^n$$

$$+ \frac{(m, aq/k, a^2 q/k; q)_\infty}{(mq, aq, a^2 q/k^2; q)_\infty} \left(\frac{ak}{k+a} \right) \sum_{n=0}^{\infty} \frac{(a, q\sqrt{a}, -q\sqrt{a}, b, c, a^2 q/bcm, \rho_1, \rho_2; q)_n (k, kq; q^2)_n}{(q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bcm/a, aq/\rho_1, aq/\rho_2; q)_n (a^2 q/k, a^2 q^2/k; q^2)_n} \left(\frac{aq^2}{k} \right)^n$$

where $m = k\rho_1 \rho_2 / aq$

$$(3.6) \sum_{n=0}^{\infty} \frac{(k, k/m, mq/\rho_1, mq/\rho_2, \rho_1, \rho_2; q)_n}{(q, mq, aq/\rho_1, aq/\rho_2, kq/\rho_1, kq/\rho_2; q)_n} \left(\frac{aq}{\rho_1 \rho_2} \right)^n {}_8\Phi_7 \left[\begin{matrix} m, aq/bc, mb/a, mc/a, \rho_1, \rho_2, kq^n, q^{-n}; q; q \\ aq/b, aq/c, mbc/a, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{1+n} \end{matrix} \right]_n$$

$$- k \sum_{n=0}^{\infty} \frac{(k, k/m, mq/\rho_1, mq/\rho_2, \rho_1, \rho_2; q)_n}{(q, mq, aq/\rho_1, aq/\rho_2, kq/\rho_1, kq/\rho_2; q)_n} \left(\frac{aq^3}{\rho_1 \rho_2} \right)^n {}_8\Phi_7 \left[\begin{matrix} m, aq/bc, mb/a, mc/a, \rho_1, \rho_2, kq^n, q^{-n}; q; q \\ aq/b, aq/c, mbc/a, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{1+n} \end{matrix} \right]_n$$

$$- m \sum_{n=0}^{\infty} \frac{(k, k/m, mq/\rho_1, mq/\rho_2, \rho_1, \rho_2; q)_n}{(q, mq, aq/\rho_1, aq/\rho_2, kq/\rho_1, kq/\rho_2; q)_n} \left(\frac{aq}{\rho_1 \rho_2} \right)^n {}_8\Phi_7 \left[\begin{matrix} m, aq/bc, mb/a, mc/a, \rho_1, \rho_2, kq^n, q^{-n}; q; q^3 \\ aq/b, aq/c, mbc/a, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{1+n} \end{matrix} \right]_n$$

$$\begin{aligned}
 & +mk \sum_{n=0}^{\infty} \frac{(k,k/m,mq/\rho_1,mq/\rho_2,\rho_1,\rho_2;q)_n}{(q,mq,aq/\rho_1,aq/\rho_2,kq/\rho_1,kq/\rho_2;q)_n} \left(\frac{aq^3}{\rho_1\rho_2}\right)^n {}_8\Phi_7 \left[\begin{matrix} m,aq/bc,mb/a,mc/a,\rho_1,\rho_2,kq^n,q^{-n};q;q^3 \\ aq/b,aq/c,mbc/a,mq/\rho_1,mq/\rho_2,mq^{1-n}/k,mq^{1+n} \end{matrix} \right]_n \\
 & = \frac{(m,k,kq/\rho_1\rho_2,aq/\rho_1,aq/\rho_2;q)_\infty}{(mq,aq,aq/\rho_1\rho_2,kq/\rho_1,kq/\rho_2;q)_\infty} {}_{10}\Phi_9 \left[\begin{matrix} a,q\sqrt{a},-q\sqrt{a},b,c,\rho_1,\rho_2,\rho_1,\rho_2,a^2q/bcm;q;kq/\rho_1\rho_2 \\ \sqrt{a},-\sqrt{a},aq/b,qa/c,qa/p_1,qa/\rho_2,qa/p_1,qa/\rho_2,bcm/a \end{matrix} \right]
 \end{aligned}$$

where $m = k\rho_1\rho_2/aq$

4. Proof

(1) By using WP Bailey's Pairs (2.10) in(2.12) theorem 1, we get

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \left\{ \frac{(k,k/m,mq/\rho_1,mq/\rho_2;q)_n}{(q,mq,aq/\rho_1,aq/\rho_2;q)_n} \right\} \left(\frac{a^2q}{k^2}\right)^n \\
 & = \frac{(aq/k,a^2q/k;q)_\infty}{(aq,a^2q/k^2;q)_\infty} \sum_{n=0}^{\infty} \frac{(k;q)_{2n}}{(a^2q/k;q)_{2n}} \left\{ \frac{(a,q\sqrt{a},-q\sqrt{a},a/k,\rho_1,\rho_2;q)_n}{(q,\sqrt{a},-\sqrt{a},kq,qa/\rho_1,qa/\rho_2;q)_n} \left(\frac{k}{a}\right)^n \right\} \left(\frac{a^2q}{k^2}\right)^n \\
 & \quad \text{where } m = k\rho_1\rho_2/aq \\
 (4.1) \quad {}_4\Phi_3 \left[\begin{matrix} k,k/m,mq/\rho_1,mq/\rho_2;q;a^2q/k^2 \\ mq,aq/\rho_1,aq/\rho_2 \end{matrix} \right] = \frac{[aq/k, a^2q/k;q]_\infty}{[aq, a^2q/k^2;q]_\infty} \sum_{n=0}^{\infty} \frac{(a,q\sqrt{a},-q\sqrt{a},a/k,\rho_1,\rho_2;q)_n (k,kq;q^2)_n}{(q,\sqrt{a},-\sqrt{a},kq,qa/\rho_1,qa/\rho_2;q)_n (a^2q/k,a^2q^2/k;q^2)_n} \left(\frac{aq}{k}\right)^n \\
 & \quad \text{where } m = k\rho_1\rho_2/aq
 \end{aligned}$$

(2) By using WP Bailey's Pairs (2.10) in (2.13) theorem 2, we get

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \left\{ \frac{(k,k/m,mq/\rho_1,mq/\rho_2;q)_n}{(q,mq,aq/\rho_1,aq/\rho_2;q)_n} \right\} \left(\frac{a^2}{k^2}\right)^n = \frac{(aq/k, a^2q/k;q)_\infty}{(aq,a^2q/k^2;q)_\infty} \left(\frac{k}{k+a}\right) \\
 & \quad \times \sum_{n=0}^{\infty} \frac{(k;q)_{2n}}{(a^2q/k;q)_{2n}} (1+aq^{2n}) \left\{ \frac{(a,q\sqrt{a},-q\sqrt{a},a/k,\rho_1,\rho_2;q)_n}{(q,\sqrt{a},-\sqrt{a},kq,qa/\rho_1,qa/\rho_2;q)_n} \left(\frac{k}{a}\right)^n \right\} \left(\frac{a^2}{k^2}\right)^n \\
 & \quad \text{where } m = k\rho_1\rho_2/aq \\
 (4.2) \quad {}_4\Phi_3 \left[\begin{matrix} k,k/m,mq/\rho_1,mq/\rho_2;q;a^2/k^2 \\ mq,aq/\rho_1,aq/\rho_2 \end{matrix} \right] \\
 & = \frac{(aq/k, a^2q/k;q)_\infty}{(aq, a^2q/k^2;q)_\infty} \left(\frac{k}{k+a}\right) \sum_{n=0}^{\infty} \frac{(a,q\sqrt{a},-q\sqrt{a},a/k,\rho_1,\rho_2;q)_n (k,kq;q^2)_n}{(q,\sqrt{a},-\sqrt{a},kq,qa/\rho_1,qa/\rho_2;q)_n (a^2q/k,a^2q^2/k;q^2)_n} \left(\frac{a}{k}\right)^n \\
 & \quad + \frac{(aq/k, a^2q/k;q)_\infty}{(aq, a^2q/k^2;q)_\infty} \left(\frac{ak}{k+a}\right) \sum_{n=0}^{\infty} \frac{(a,q\sqrt{a},-q\sqrt{a},a/k,\rho_1,\rho_2;q)_n (k,kq;q^2)_n}{(q,\sqrt{a},-\sqrt{a},kq,qa/\rho_1,qa/\rho_2;q)_n (a^2q/k,a^2q^2/k;q^2)_n} \left(\frac{a^2q^2}{k^2}\right)^n \\
 & \quad \text{where } m = k\rho_1\rho_2/aq
 \end{aligned}$$

(3) By using WP Bailey's Pairs (2.10) in (2.14) theorem 3, we get

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \left(\frac{1-kq^{2n}}{1-k} \right) \frac{(\rho_1,\rho_2;q)_n}{(kq/\rho_1,kq/\rho_2;q)_n} \left\{ \frac{(k,k/m,mq/\rho_1,mq/\rho_2;q)_n}{(q,mq,aq/\rho_1,aq/\rho_2;q)_n} \right\} \left(\frac{aq}{\rho_1\rho_2}\right)^n \\
 & = \frac{(kq,kq/\rho_1\rho_2,aq/\rho_1,aq/\rho_2;q)_\infty}{(aq,aq/\rho_1\rho_2,kq/\rho_1,kq/\rho_2;q)_\infty} \sum_{n=0}^{\infty} \frac{(\rho_1,\rho_2;q)_n}{(aq/\rho_1,aq/\rho_2;q)_n} \left\{ \frac{(a,q\sqrt{a},-q\sqrt{a},a/k,\rho_1,\rho_2;q)_n}{(q,\sqrt{a},-\sqrt{a},kq,qa/\rho_1,qa/\rho_2;q)_n} \left(\frac{k}{a}\right)^n \right\} \left(\frac{aq}{\rho_1\rho_2}\right)^n \\
 & \quad \text{where } m = k\rho_1\rho_2/aq
 \end{aligned}$$

$$\begin{aligned}
 (4.3) \quad {}_6\Phi_5 \left[\begin{matrix} k,k/m,mq/\rho_1,mq/\rho_2,\rho_1,\rho_2;q;aq/\rho_1\rho_2 \\ mq,kq/\rho_1,kq/\rho_2,aq/\rho_1,aq/\rho_2 \end{matrix} \right] - k {}_6\Phi_5 \left[\begin{matrix} k,k/m,mq/\rho_1,mq/\rho_2,\rho_1,\rho_2;q;aq^3/\rho_1\rho_2 \\ mq,kq/\rho_1,kq/\rho_2,aq/\rho_1,aq/\rho_2 \end{matrix} \right] \\
 & = \frac{(k,kq/\rho_1\rho_2,aq/\rho_1,aq/\rho_2;q)_\infty}{(aq,aq/\rho_1\rho_2,kq/\rho_1,kq/\rho_2;q)_\infty} {}_8\Phi_7 \left[\begin{matrix} a,q\sqrt{a},-q\sqrt{a},a/k,\rho_1,\rho_2,\rho_1,\rho_2;q;kq/\rho_1\rho_2 \\ \sqrt{a},-\sqrt{a},kq,qa/\rho_1,qa/\rho_2,qa/\rho_1,qa/\rho_2 \end{matrix} \right] \\
 & \quad \text{where } m = k\rho_1\rho_2/aq
 \end{aligned}$$

(4) By using WP Bailey's Pairs (2.11) in (2.12) theorem 1, we get

$$\begin{aligned} \sum_{n=0}^{\infty} & \left\{ \frac{(k,k/m,mq/\rho_1,mq/\rho_2;q)_n}{(q,mq,aq/\rho_1,aq/\rho_2;q)_n} \sum_{r=0}^{\infty} \left(\frac{1-mq^{2r}}{1-m} \right) \frac{(m,mb/a,mc/a,aq/bc,\rho_1,\rho_2,kq^n,q^{-n};q)_r q^r}{(q,aq/b,aq/c,mbc/a,mq/\rho_1,mq/\rho_2,mq^{1-n}/k,mq^{1+n};q)_r} \right\} \left(\frac{a^2 q}{k^2} \right)^n \\ & = \frac{(aq/k, a^2 q/k;q)_\infty}{(aq, a^2 q/k^2;q)_\infty} \sum_{n=0}^{\infty} \frac{(k;q)_{2n}}{(a^2 q/k;q)_{2n}} \left\{ \frac{(a,q\sqrt{a},-q\sqrt{a},b,c,a^2 q/bcm,\rho_1,\rho_2;q)_n}{(q,\sqrt{a},-\sqrt{a},aq/b,aq/c,bcm/a,aq/\rho_1,aq/\rho_2;q)_n} \left(\frac{k}{a} \right)^n \right\} \left(\frac{a^2 q}{k^2} \right)^n \\ & \quad \text{where } m = k\rho_1\rho_2/aq \end{aligned}$$

$$(4.4) \sum_{n=0}^{\infty} \frac{(k,k/m,mq/\rho_1,mq/\rho_2;q)_n}{(q,mq,aq/\rho_1,aq/\rho_2;q)_n} \left(\frac{a^2 q}{k^2} \right)^n {}_8\Phi_7 \left[\begin{matrix} m, aq/bc, mb/a, mc/a, \rho_1, \rho_2, kq^n, q^{-n}; q \\ aq/b, aq/c, mbc/a, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{1+n} \end{matrix} \right]_n$$

$$-m \sum_{n=0}^{\infty} \frac{(k,k/m,mq/\rho_1,mq/\rho_2;q)_n}{(q,mq,aq/\rho_1,aq/\rho_2;q)_n} \left(\frac{a^2 q}{k^2} \right)^n {}_8\Phi_7 \left[\begin{matrix} m, aq/bc, mb/a, mc/a, \rho_1, \rho_2, kq^n, q^{-n}; q \\ aq/b, aq/c, mbc/a, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{1+n} \end{matrix} \right]_n$$

$$= \frac{(m,aq/k,a^2 q/k;q)_\infty}{(mq,aq,a^2 q/k^2;q)_\infty} \sum_{n=0}^{\infty} \frac{(a,q\sqrt{a},-q\sqrt{a},b,c,a^2 q/bcm,\rho_1,\rho_2;q)_n (k,kq;q^2)_n}{(q,\sqrt{a},-\sqrt{a},aq/b,aq/c,bcm/a,aq/\rho_1,aq/\rho_2;q)_n (a^2 q/k,a^2 q^2/k;q^2)_n} \left(\frac{aq}{k} \right)^n$$

$$\quad \text{where } m = k\rho_1\rho_2/aq$$

(5) By using WP Bailey's Pairs (2.11) in (2.13) theorem 2, we get

$$\begin{aligned} \sum_{n=0}^{\infty} & \left\{ \frac{(k,k/m,mq/\rho_1,mq/\rho_2;q)_n}{(q,mq,aq/\rho_1,aq/\rho_2;q)_n} \sum_{r=0}^{\infty} \left(\frac{1-mq^{2r}}{1-m} \right) \frac{(m,mb/a,mc/a,aq/bc,\rho_1,\rho_2,kq^n,q^{-n};q)_r q^r}{(q,aq/b,aq/c,mbc/a,mq/\rho_1,mq/\rho_2,mq^{1-n}/k,mq^{1+n};q)_r} \right\} \left(\frac{a^2}{k^2} \right)^n \\ & = \frac{(aq/k, a^2 q/k;q)_\infty}{(aq, a^2 q/k^2;q)_\infty} \left(\frac{k}{k+a} \right) \sum_{n=0}^{\infty} \frac{(k;q)_{2n}}{(a^2 q/k;q)_{2n}} (1 + aq^{2n}) \left\{ \frac{(a,q\sqrt{a},-q\sqrt{a},b,c,a^2 q/bcm,\rho_1,\rho_2;q)_n}{(q,\sqrt{a},-\sqrt{a},aq/b,aq/c,bcm/a,aq/\rho_1,aq/\rho_2;q)_n} \left(\frac{k}{a} \right)^n \right\} \left(\frac{a^2}{k^2} \right)^n \\ & \quad \text{where } m = k\rho_1\rho_2/aq \end{aligned}$$

$$(4.5) \sum_{n=0}^{\infty} \frac{(k,k/m,mq/\rho_1,mq/\rho_2;q)_n}{(q,mq,aq/\rho_1,aq/\rho_2;q)_n} \left(\frac{a^2}{k^2} \right)^n {}_8\Phi_7 \left[\begin{matrix} m, aq/bc, mb/a, mc/a, \rho_1, \rho_2, kq^n, q^{-n}; q \\ aq/b, aq/c, mbc/a, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{1+n} \end{matrix} \right]_n$$

$$-m \sum_{n=0}^{\infty} \frac{(k,k/m,mq/\rho_1,mq/\rho_2;q)_n}{(q,mq,aq/\rho_1,aq/\rho_2;q)_n} \left(\frac{a^2}{k^2} \right)^n {}_8\Phi_7 \left[\begin{matrix} m, aq/bc, mb/a, mc/a, \rho_1, \rho_2, kq^n, q^{-n}; q \\ aq/b, aq/c, mbc/a, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{1+n} \end{matrix} \right]_n$$

$$= \frac{(m,aq/k,a^2 q/k;q)_\infty}{(mq,aq,a^2 q/k^2;q)_\infty} \left(\frac{k}{k+a} \right) \sum_{n=0}^{\infty} \frac{(a,q\sqrt{a},-q\sqrt{a},b,c,a^2 q/bcm,\rho_1,\rho_2;q)_n (k,kq;q^2)_n}{(q,\sqrt{a},-\sqrt{a},aq/b,aq/c,bcm/a,aq/\rho_1,aq/\rho_2;q)_n (a^2 q/k,a^2 q^2/k;q^2)_n} \left(\frac{a}{k} \right)^n$$

$$+ \frac{(m,aq/k,a^2 q/k;q)_\infty}{(mq,aq,a^2 q/k^2;q)_\infty} \left(\frac{ak}{k+a} \right) \sum_{n=0}^{\infty} \frac{(a,q\sqrt{a},-q\sqrt{a},b,c,a^2 q/bcm,\rho_1,\rho_2;q)_n (k,kq;q^2)_n}{(q,\sqrt{a},-\sqrt{a},aq/b,aq/c,bcm/a,aq/\rho_1,aq/\rho_2;q)_n (a^2 q/k,a^2 q^2/k;q^2)_n} \left(\frac{aq^2}{k} \right)^n$$

$$\quad \text{where } m = k\rho_1\rho_2/aq$$

(6) By using WP Bailey's Pairs (2.11) in (2.14) theorem 3, we get

$$\begin{aligned} \sum_{n=0}^{\infty} & \left(\frac{1-kq^{2n}}{1-k} \right) \frac{(\rho_1,\rho_2;q)_n}{(kq/\rho_1,kq/\rho_2;q)_n} \left(\frac{aq}{\rho_1\rho_2} \right)^n \\ & \times \left\{ \frac{(k,k/m,mq/\rho_1,mq/\rho_2;q)_n}{(q,mq,aq/\rho_1,aq/\rho_2;q)_n} \sum_{r=0}^{\infty} \left(\frac{1-mq^{2r}}{1-m} \right) \frac{(m,mb/a,mc/a,aq/bc,\rho_1,\rho_2,kq^n,q^{-n};q)_r q^r}{(q,aq/b,aq/c,mbc/a,mq/\rho_1,mq/\rho_2,mq^{1-n}/k,mq^{1+n};q)_r} \right\} \\ & = \frac{(kq,kq/\rho_1\rho_2,aq/\rho_1,aq/\rho_2;q)_\infty}{(aq,aq/\rho_1\rho_2,kq/\rho_1,kq/\rho_2;q)_\infty} \sum_{n=0}^{\infty} \frac{(\rho_1,\rho_2;q)_n}{(aq/\rho_1,aq/\rho_2;q)_n} \left\{ \frac{(a,q\sqrt{a},-q\sqrt{a},b,c,a^2 q/bcm,\rho_1,\rho_2;q)_n}{(q,\sqrt{a},-\sqrt{a},aq/b,aq/c,bcm/a,aq/\rho_1,aq/\rho_2;q)_n} \left(\frac{k}{a} \right)^n \right\} \left(\frac{aq}{\rho_1\rho_2} \right)^n \\ & \quad \text{where } m = k\rho_1\rho_2/aq \end{aligned}$$

$$(4.6) \sum_{n=0}^{\infty} \frac{(1-kq^{2n})(k,k/m,mq/\rho_1,mq/\rho_2,\rho_1,\rho_2;q)_n}{(q,mq,aq/\rho_1,aq/\rho_2,kq/\rho_1,kq/\rho_2;q)_n} \left(\frac{aq}{\rho_1\rho_2} \right)^n {}_8\Phi_7 \left[\begin{matrix} m, aq/bc, mb/a, mc/a, \rho_1, \rho_2, kq^n, q^{-n}; q \\ aq/b, aq/c, mbc/a, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{1+n} \end{matrix} \right]_n$$

$$-m \sum_{n=0}^{\infty} \frac{(1-kq^{2n})(k,k/m,mq/\rho_1,mq/\rho_2,\rho_1,\rho_2;q)_n}{(q,mq,aq/\rho_1,aq/\rho_2,kq/\rho_1,kq/\rho_2;q)_n} \left(\frac{aq}{\rho_1\rho_2} \right)^n {}_8\Phi_7 \left[\begin{matrix} m, aq/bc, mb/a, mc/a, \rho_1, \rho_2, kq^n, q^{-n}; q \\ aq/b, aq/c, mbc/a, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{1+n} \end{matrix} \right]_n$$

$$= \frac{(m,k,kq/\rho_1\rho_2,aq/\rho_1,aq/\rho_2;q)_\infty}{(mq,aq,aq/\rho_1\rho_2,kq/\rho_1,kq/\rho_2;q)_\infty} {}_{10}\Phi_9 \left[\begin{matrix} a,q\sqrt{a},-q\sqrt{a},b,c,\rho_1,\rho_2,a^2 q/bcm;q \\ \sqrt{a},-\sqrt{a},aq/b,aq/c,aq/\rho_1,aq/\rho_2,aq/\rho_1,aq/\rho_2,bcm/a \end{matrix} \right]$$

$$\quad \text{where } m = k\rho_1\rho_2/aq$$

$$(4.7) \sum_{n=0}^{\infty} \frac{(k,k/m,mq/\rho_1,mq/\rho_2,\rho_1,\rho_2;q)_n}{(q,mq,aq/\rho_1,aq/\rho_2,kq/\rho_1,kq/\rho_2;q)_n} \left(\frac{aq}{\rho_1\rho_2} \right)^n {}_8\Phi_7 \left[\begin{matrix} m, aq/bc, mb/a, mc/a, \rho_1, \rho_2, kq^n, q^{-n}; q \\ aq/b, aq/c, mbc/a, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{1+n} \end{matrix} \right]_n$$

$$\begin{aligned}
 & -k \sum_{n=0}^{\infty} \frac{(k, k/m, mq/\rho_1, mq/\rho_2, \rho_1, \rho_2; q)_n}{(q, mq, aq/\rho_1, aq/\rho_2, kq/\rho_1, kq/\rho_2; q)_n} \left(\frac{aq^3}{\rho_1 \rho_2}\right)^n {}_8\Phi_7 \left[\begin{matrix} m, aq/bc, mb/a, mc/a, \rho_1, \rho_2, kq^n, q^{-n}; q; q \\ aq/b, aq/c, mbc/a, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{1+n} \end{matrix} \right]_n \\
 & -m \sum_{n=0}^{\infty} \frac{(k, k/m, mq/\rho_1, mq/\rho_2, \rho_1, \rho_2; q)_n}{(q, mq, aq/\rho_1, aq/\rho_2, kq/\rho_1, kq/\rho_2; q)_n} \left(\frac{aq}{\rho_1 \rho_2}\right)^n {}_8\Phi_7 \left[\begin{matrix} m, aq/bc, mb/a, mc/a, \rho_1, \rho_2, kq^n, q^{-n}; q; q^3 \\ aq/b, aq/c, mbc/a, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{1+n} \end{matrix} \right]_n \\
 & +mk \sum_{n=0}^{\infty} \frac{(k, k/m, mq/\rho_1, mq/\rho_2, \rho_1, \rho_2; q)_n}{(q, mq, aq/\rho_1, aq/\rho_2, kq/\rho_1, kq/\rho_2; q)_n} \left(\frac{aq^3}{\rho_1 \rho_2}\right)^n {}_8\Phi_7 \left[\begin{matrix} m, aq/bc, mb/a, mc/a, \rho_1, \rho_2, kq^n, q^{-n}; q; q^3 \\ aq/b, aq/c, mbc/a, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{1+n} \end{matrix} \right]_n \\
 & = \frac{(m, k, kq/\rho_1 \rho_2, aq/\rho_1, aq/\rho_2; q)_{\infty}}{(mq, aq, aq/\rho_1 \rho_2, kq/\rho_1, kq/\rho_2; q)_{\infty}} {}_{10}\Phi_9 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, \rho_1, \rho_2, \rho_1, \rho_2, a^2 q/bcm; q; kq/\rho_1 \rho_2 \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/\rho_1, aq/\rho_2, aq/\rho_1, aq/\rho_2, bcm/a \end{matrix} \right]
 \end{aligned}$$

where $m = k\rho_1 \rho_2 / aq$

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