A Double Expansion Formula for Generalized Multivariable Gimel-Function Involving Jacobi Polynomials and Bessel Functions

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ABSTRACT
In this paper, we present a double expansion formula for the generalized multivariable Gimel-function involving Jacobi polynomials and Bessel functions.

KEYWORDS : Multivariable Gimel-function, multiple integral contours, Jacobi polynomial, Bessel function, Double expansion serie.

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1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C}, \mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The subject of expansion formulae and Fourier series of special functions occupies a large place in the literature of special functions. Certain double expansion formulae and double Fourier series of generalized hypergeometric functions play an important rôle in the development of the theories of special functions and two- dimensional boundary value problems. In this paper, we establish a double expansion formula for generalized multivariable Gimel-function.

We define a generalized transcendental function of several complex variables.

$$\gimel(z_1,\cdots,z_r) = \gimel^{m_2,n_2;m_3,n_3;\cdots;m_r,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r \end{array}\right|$$

$$[(\mathrm{a}_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},$$
$$[(\mathrm{b}_{2j};\beta^{(1)}_{2j},\beta^{(2)}_{2j};B_{2j})]_{1,m_2},[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{m_2+1,q_{i_2}},[(b_{3j};\beta^{(1)}_{3j},\beta^{(2)}_{3j},\beta^{(3)}_{3j};B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{m_3+1,q_{i_3}};\cdots;[(\mathrm{b}_{rj};\beta^{(1)}_{rj},\cdots,\beta^{(r)}_{rj};B_{rj})_{1,m_r}],$$

$$[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{n_r+1,p_r}]:\ [(\mathrm{c}^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]$$
$$[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{m_r+1,q_r}]:[(\mathrm{d}^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]$$

$$\left.\begin{array}{l};\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})_{m^{(r)}+1,p^{(r)}_i}]\\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})_{n^{(r)}+1,q^{(r)}_i}]\end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{m_2}\Gamma^{B_{2j}}(b_{2j}-\sum_{k=1}^{2}\beta_{2j}^{(k)}s_k)\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha_{2ji_2}^{(k)}s_k)\prod_{j=m_2+1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta_{2ji2}^{(k)}s_k)]}$$

$$\frac{\prod_{j=1}^{m_3}\Gamma^{B_{3j}}(b_{3j}-\sum_{k=1}^{3}\beta_{3j}^{(k)}s_k)\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=m_3+1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{m_r}\Gamma^{B_{rj}}(b_{rj}-\sum_{k=1}^{r}\beta_{rj}^{(k)}s_k)\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=m_r+1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \quad (1.2)$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \quad (1.3)$$

1) $[(c_j^{(1)};\gamma_j^{(1)})]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $m_2,n_2,\cdots,m_r,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2\leqslant q_{i_2},0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant m_r\leqslant q_{i_r},0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$

$0\leqslant n^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant p_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kj}^{(l)},B_{kj}\in\mathbb{R}^+;(j=1,\cdots,m_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)} \in \mathbb{R}^+; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r).$

5) $c_j^{(k)} \in \mathbb{C}; (j = 1, \cdots, n_k); (k = 1, \cdots, r); d_j^{(k)} \in \mathbb{C}; (j = 1, \cdots, m_k); (k = 1, \cdots, r).$

$a_{kji_k} \in \mathbb{C}; (j = n_k + 1, \cdots, p_{i_k}); (k = 2, \cdots, r).$

$b_{kji_k} \in \mathbb{C}; (j = m_k + 1, \cdots, q_{i_k}); (k = 2, \cdots, r).$

$d_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = m^{(k)} + 1, \cdots, q_{i^{(k)}}); (k = 1, \cdots, r).$

$\gamma_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r).$

The contour L_k is in the $s_k (k = 1, \cdots, r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1 - a_{2j} + \sum_{k=1}^{2} \alpha_{2j}^{(k)} s_k\right) (j = 1, \cdots, n_2), \Gamma^{A_{3j}}\left(1 - a_{3j} + \sum_{k=1}^{3} \alpha_{3j}^{(k)} s_k\right)$ $(j = 1, \cdots, n_3), \cdots, \Gamma^{A_{rj}}\left(1 - a_{rj} + \sum_{i=1}^{r} \alpha_{rj}^{(i)}\right) (j = 1, \cdots, n_r)$, $\Gamma^{C_j^{(k)}}\left(1 - c_j^{(k)} + \gamma_j^{(k)} s_k\right) (j = 1, \cdots, n^{(k)})(k = 1, \cdots, r)$to the right of the contour L_k and the poles of $\Gamma^{B_{2j}}\left(b_{2j} - \sum_{k=1}^{2} \beta_{2j}^{(k)} s_k\right) (j = 1, \cdots, m_2), \Gamma^{B_{3j}}\left(b_{3j} - \sum_{k=1}^{3} \beta_{3j}^{(k)} s_k\right) (j = 1, \cdots, m_3)$ $, \cdots, \Gamma^{B_{rj}}\left(b_{rj} - \sum_{i=1}^{r} \beta_{rj}^{(i)}\right) (j = 1, \cdots, m_r)$, $\Gamma^{D_j^{(k)}}\left(d_j^{(k)} - \delta_j^{(k)} s_k\right) (j = 1, \cdots, m^{(k)})(k = 1, \cdots, r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2} A_i^{(k)} \pi$ where

$$A_i^{(k)} = \sum_{j=1}^{m^{(k)}} D_j^{(k)} \delta_j^{(k)} + \sum_{j=1}^{n^{(k)}} C_j^{(k)} \gamma_j^{(k)} - \tau_{i^{(k)}} \left(\sum_{j=m^{(k)}+1}^{q_i^{(k)}} D_{ji^{(k)}}^{(k)} \delta_{ji^{(k)}}^{(k)} + \sum_{j=n^{(k)}+1}^{p_i^{(k)}} C_{ji^{(k)}}^{(k)} \gamma_{ji^{(k)}}^{(k)} \right) +$$

$$\sum_{j=1}^{n_2} A_{2j} \alpha_{2j}^{(k)} + \sum_{j=1}^{m_2} B_{2j} \beta_{2j}^{(k)} - \tau_{i_2} \left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2} \alpha_{2ji_2}^{(k)} + \sum_{j=m_2+1}^{q_{i_2}} B_{2ji_2} \beta_{2ji_2}^{(k)} \right) + \cdots +$$

$$\sum_{j=1}^{n_r} A_{rj} \alpha_{rj}^{(k)} + \sum_{j=1}^{m_r} B_{rj} \beta_{rj}^{(k)} - \tau_{i_r} \left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r} \alpha_{rji_r}^{(k)} + \sum_{j=m_r+1}^{q_{i_r}} B_{rji_r} \beta_{rji_r}^{(k)} \right) \tag{1.4}$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \rightarrow 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \rightarrow \infty$ where $i = 1, \cdots, r :$

$\alpha_i = \min_{\substack{1 \leqslant j \leqslant m_i \\ 1 \leqslant j \leqslant m^{(i)}}} Re \left(\sum_{h=2}^{r} \sum_{h'=1}^{h} B_{hj} \frac{b_{hj}}{\beta_{hj}^{h'}} + D_j^{(i)} \frac{d_j^{(i)}}{\delta_j^{(i)}} \right)$ and $\beta_i = \max_{\substack{1 \leqslant j \leqslant n_i \\ 1 \leqslant j \leqslant n^{(i)}}} Re \left(\sum_{h=2}^{r} \sum_{h'=1}^{h} A_{hj} \frac{a_{hj} - 1}{\alpha_{hj}^{h'}} + C_j^{(i)} \frac{c_j^{(i)} - 1}{\gamma_j^{(i)}} \right)$

Remark 1.
If $m_2 = n_2 = \cdots = m_{r-1} = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the generalized multivariable Gimel-function reduces in the generalized multivariable Aleph- function (extension of multivariable Aleph-function defined by Ayant [1]).

Remark 2.

If $m_2=n_2=\cdots=m_r=n_r=p_{i_2}=q_{i_2}=\cdots=p_{i_r}=q_{i_r}=0$ and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2=$ $=\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then the generalized multivariable Gimel-function reduces in a generalized multivariable I-function (extension of multivariable I-function defined by Prathima et al. [7]).

Remark 3.

If $A_{2j}=B_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=B_{rj}=A_{rji_r}=B_{rji_r}=1$ and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2$ $=\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then the generalized multivariable Gimel-function reduces in generalized of multivariable I-function (extension of multivariable I-function defined by Prasad [6]).

Remark 4.

If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the generalized multivariable H-function (extension of multivariable H-function defined by Srivastava and panda [8,9]).

In your investigation, we shall use the following notations.

$$\mathbb{A}=[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.5}$$

$$\mathbf{A}=[(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A=[(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \tag{1.7}$$

$$\mathbb{B}=[(b_{2j};\beta_{2j}^{(1)},\beta_{2j}^{(2)};B_{2j})]_{1,m_2},[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{m_2+1,q_{i_2}},[(b_{3j};\beta_{3j}^{(1)},\beta_{3j}^{(2)},\beta_{3j}^{(3)};B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{m_3+1,q_{i_3}};\cdots;[(\mathrm{b}_{(r-1)j};\beta_{(r-1)j}^{(1)},\cdots,\beta_{(r-1)j}^{(r-1)};B_{(r-1)j})_{1,m_{r-1}}],$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})_{m_{r-1}+1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B}=[(\mathrm{b}_{rj};\beta_{rj}^{(1)},\cdots,\beta_{rj}^{(r)};B_{rj})_{1,m_r}],[\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{m_r+1,q_{i_r}}] \tag{1.9}$$

$$\mathrm{B}=[(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}];\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \tag{1.10}$$

$$U=m_2,n_2;m_3,n_3;\cdots;m_{r-1},n_{r-1};V=m^{(1)};n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X=p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

2. Required results.

In this section, we give four formulae. These results will be used in the following sections.

Lemma 1. ([4], p. 1240, Eq. 4)

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\beta}P_m^{(\alpha,\beta)}(x)\mathrm{d}x=\frac{2^{\beta+\rho+1}\Gamma(\rho+1)\Gamma(\beta+n+1)\Gamma(\alpha-\rho+n)}{n!\Gamma(\alpha-\rho)\Gamma(\beta+\rho+n+2)} \tag{2.1}$$

provided $Re(\rho)>-1, Re(\beta)>-1.$

Lemma 2. (see Luke,[5])

$$\int_{0}^{\infty}y^{\sigma-1}\cos yJ_v(x)\mathrm{d}y=\frac{2^{\sigma-1}\sqrt{\pi}\Gamma\left(\frac{1}{2}-\sigma\right)\Gamma\left(\frac{v+\sigma}{2}\right)}{\Gamma\left(1-\frac{v-\sigma}{2}\right)\Gamma\left(\frac{1-v-\sigma}{2}\right)\Gamma\left(\frac{1+v-\sigma}{2}\right)} \tag{2.2}$$

provided $Re(\sigma+v)>0.$

The orthogonality property of the Jacobi polynomials [3].

Lemma 3.

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)\mathrm{d}x=\left[\begin{array}{c}0 \text{ if m} \neq n\\ .\\ \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)} \text{ if m = n}\end{array}\right. \tag{2.3}$$

provided $Re(\alpha)>-1, Re(\beta)>-1.$

Orthogonality property for Bessel's function [5]

Lemma 3.

$$\int_{0}^{\infty}x^{-1}J_{u+2m+1}(x)J_{u+2n+1}(x)\mathrm{d}x=\left[\begin{array}{c}0, \text{ if m} \neq n\\ .\\ \frac{(u+2n+1)^{-1}}{2}, \text{ if m = n}\end{array}\right. \tag{2.4}$$

provided $Re(u)+m+n>-1.$

3. Main integrals.

In this section, we establish two integrals.

Theorem 1.

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\beta}P_m^{(\alpha,\beta)}(x)\daleth\left(z_1(1-x)^{h_1},\cdots,z_r(1-x)^{h_r}\right)\mathrm{d}x=\frac{2^{\rho+\beta+1}\Gamma(1+m+\beta)}{m!}$$

$$\daleth^{U;m_r+1,n_r+1:V}_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c}2^{h_1}z_1 & \mathbb{A};\ (\text{-}\rho;h_1,\cdots,h_r;1),\mathbf{A},(\alpha-\rho;h_1,\cdots,h_r;1):A\\ \cdot & \cdot\\ \cdot & \cdot\\ 2^{h_r}z_r & \mathbb{B};\ (\alpha-\rho+m;h_1,\cdots,h_r;1),\mathbf{B},(-1-\beta-\rho-m;h_1,\cdots,h_r;1):B\end{array}\right) \tag{3.1}$$

provided

$$h_i>0(i=1,\cdots,r),\ Re(\beta)>-1, Re(\rho)+\sum_{i=1}^{r}h_i\min_{\substack{1\leqslant j\leqslant m_i\\ 1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-1.$$

$|arg(z_i(1-x)^{h_i})|<\dfrac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Proof

To prove the theorem 1, we replace the generalized multivariable Gimel-function by this multiple integrals contour with the help of (1.1), change the order of integrations which is justified under the conditions mentioned above. We get

$$\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\left[\int_{-1}^{1}(1-x)^{\rho+\sum_{i=1}^{r}h_is_i}(1+x)^{\beta}P_m^{(\alpha,\beta)}(x)\mathrm{d}x\right]\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.2}$$

Evaluate the inner integral with the help of lemma 1 and interpreting the Mellin-Barnes multiple integrals contour in terms of the multivariable Gimel-function, we get the desired result (3.1).

Theorem 2.

$$\int_0^{\infty}y^{\sigma-1}\cos y J_v(x)\gimel(z_1y^{2k_1},\cdots,z_ry^{2k_r})\mathrm{d}y=2^{\sigma-1}\sqrt{\pi}\,\gimel_{X;p_{i_r}+4,q_{i_r}+1,\tau_{i_r}:R_r:Y}^{U;m_r+1,n_r+1:V}$$

$$\left(\begin{array}{c|c} 4^{k_1}z_1 & \mathbb{A};\left(1-\frac{\sigma+v}{2};k_1,\cdots,k_r;1\right),\mathbf{A},\left(1+\frac{v-\sigma}{2};k_1,\cdots,k_r;1\right),\left(\frac{1-v-\sigma}{2};k_1,\cdots,k_r;1\right),:\left(\frac{1+v-\sigma}{2};k_1,\cdots,k_r;1\right):A \\ \cdot & \cdot \\ \cdot & \cdot \\ 4^{k_r}z_r & \mathbb{B};\left(\frac{1}{2}-\sigma;2k_1,\cdots,2k_r;1\right),\mathbf{B}:B \end{array}\right) \tag{3.3}$$

provided

$$h_i>0(i=1,\cdots,r),\ Re(\sigma)+2\sum_{i=1}^{r}k_i\min_{\substack{1\leqslant j\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0.$$

$|arg(z_i(1-x)^{h_i})|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Proof

To prove the theorem 2, we replace the generalized multivariable Gimel-function by this multiple integrals contour with the help of (1.1), change the order of integrations which is justified under the conditions mentioned above. We get

$$\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\left[\int_0^{\infty}y^{\sigma+\sum_{i=1}^{r}k_is_i-1}\cos yJ_v(x)\gimel(z_1y^{2k_1},\cdots,z_ry^{2k_r})\mathrm{d}y\right]\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.4}$$

Evaluate the inner integral with the help of lemma 2 and interpreting the Mellin-Barnes multiple integrals contour in terms of the multivariable Gimel-function, we get the desired result (3.3).

4. Double expansion formula.

The double expansion formula to be establish is

Theorem 3.

$$(1-x)^{\rho}y^{\sigma}\cos y\gimel\left(z_1(1-x)^{h_1}y_1^{2k_1},\cdots,z_r(1-x)^{h_r}y_r^{2k_r}\right)=2^{\rho+\sigma}\sqrt{\pi}$$

$$\sum_{s,t=0}^{\infty}\frac{(\alpha+\beta+2s+1)\Gamma(\alpha+\beta+s+1)(v+2t+1)}{\Gamma(\alpha+s+1)}P_s^{(\alpha,\beta)}(x)J_{v+2t+1}(y)$$

$$\gimel_{X;p_{i_r}+6,q_{i_r}+3,\tau_{i_r}:R_r:Y}^{U;m_r+2,n_r+3:V}\left(\begin{array}{c|c} 2^{h_1+2k_r}z_1 & \mathbb{A};\mathrm{A}_1,\mathbf{A},A_2:A \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{h_r+2k_r}z_r & \mathbb{B};\mathrm{B}_1,\mathbf{B},B_2:B \end{array}\right) \tag{4.1}$$

where

$$A_1 = (-\rho-\alpha; h_1,\cdots,h_r;1), \left(1-\frac{\sigma+\upsilon+2t+1}{2}; k_1,\cdots,k_r;1\right) \tag{4.2}$$

$$A_2 = (-\rho; h_1,\cdots,h_r;1), \left(1+\frac{\upsilon+2t+1-\sigma}{2}; k_1,\cdots,k_r;1\right), \left(-\frac{\upsilon+2t+\sigma}{2}; k_1,\cdots,k_r;1\right), \left(1+\frac{\upsilon+2t-\sigma}{2}; k_1,\cdots,k_r;1\right) \tag{4.3}$$

$$B_1 = \left(\frac{1}{2}-\sigma; 2k_1,\cdots,2k_r;1\right), (-\rho+s; h_1,\cdots,h_r;1); B_2 = (-1-\alpha-\beta-\rho-s; h_1,\cdots,h_r;1) \tag{4.4}$$

valid under the existence conditions mentioned in (3.1) and (3.3).

Proof

To establish (4.1), let

$$f(x,y) = (1-x)^\rho y^\sigma \cos y \daleth \left(z_1(1-x)^{h_1} y_1^{2k_1},\cdots,z_r(1-x)^{h_r} y_r^{2k_r}\right) = \sum_{s,t=0}^{\infty} A_{s,t} P_s^{(\alpha,\beta)}(x) J_{\upsilon+2t+1}(y) \tag{4.5}$$

The above equation is valid, since $f(x,y)$ is continuous and bounded variation in the region $(-1,1)\times(0,\infty)$. Multiplying both sides of (4.5) by $y^{-1}J_{\upsilon+2v+1}(y)$, integrating with respect to y from 1 to ∞ and using the (2.4) and (3.3), we obtain

$$2^{\sigma-1}(1-x)^\rho \sqrt{\pi} \daleth^{U;m_r+1,n_r+1:V}_{X;p_{i_r}+4,q_{i_r}+1,\tau_{i_r}:R_r:Y} \left(\begin{array}{c|c} 4^{k_1}(1-x)^{h_1} z_1 & \mathbb{A};\ \mathrm{C}_1\mathbf{A}, C_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ 4^{k_r}(1-x)^{h_r} z_r & \mathbb{B};\ \mathrm{D}_1, \mathbf{B} : B \end{array} \right) =$$

$$\sum_{s=0}^{\infty} A_{s,v}(4v+2\upsilon+2)^{-1} P_s^{(\alpha,\beta)}(x) \tag{4.6}$$

where

$$C_1 = \left(1-\frac{\sigma+\upsilon+2v+1}{2}; k_1,\cdots,k_r;1\right); D_1 = \left(\frac{1}{2}-\sigma; 2k_1,\cdots,2k_r:1\right) \tag{4.7}$$

$$C_2 = \left(1+\frac{\upsilon+2v+1-\sigma}{2}; k_1,\cdots,k_r;1\right), \left(-\frac{\upsilon+2v+\sigma}{2}; k_1,\cdots,k_r;1\right), \left(1+\frac{\upsilon+2v-\sigma}{2}; k_1,\cdots,k_r;1\right) \tag{4.8}$$

Multiplying both sides of (4.6) by $(1-x)^\alpha(1+x)^\beta P_u^{(\alpha,\beta)}(x)$, integrating with respect to x from -1 to 1 and using (2.3) and (3.1), we obtain

$$A_{u,v} = \frac{2^{\rho+\sigma}\sqrt{\pi}(\alpha+\beta+2u+1)\Gamma(\alpha+\beta+u+1)(\upsilon+2v+1)}{\Gamma(\alpha+u+1)}$$

$$\daleth^{U;m_r+2,n_r+3:V}_{X;p_{i_r}+6,q_{i_r}+3,\tau_{i_r}:R_r:Y} \left(\begin{array}{c|c} 2^{h_1+2k_r} z_1 & \mathbb{A};\ \mathrm{A}'_1, \mathbf{A}, A'_2 : A \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{h_r+2k_r} z_r & \mathbb{B};\ \mathrm{B}'_1, \mathbf{B}, B'_2 : B \end{array} \right) \tag{4.9}$$

where

$$A'_1 = (-\rho-\alpha; h_1,\cdots,h_r;1), \left(1-\frac{\sigma+\upsilon+2v+1}{2}; k_1,\cdots,k_r;1\right) \tag{4.10}$$

$$A'_2 = (-\rho; h_1,\cdots,h_r;1), \left(1+\frac{\upsilon+2v+1-\sigma}{2}; k_1,\cdots,k_r;1\right), \left(-\frac{\upsilon+2v+\sigma}{2}; k_1,\cdots,k_r;1\right), \left(1+\frac{\upsilon+2v-\sigma}{2}; k_1,\cdots,k_r;1\right) \tag{4.11}$$

$$B_1' = \left(\frac{1}{2} - \sigma; 2k_1, \cdots, 2k_r; 1\right), (-\rho + u; h_1, \cdots, h_r; 1); B_2 = (-1 - \alpha - \beta - \rho - u; h_1, \cdots, h_r; 1) \tag{4.12}$$

Finally substituting the value of $A_{s,t}$ in (4.5), we obtain the desired result (4.1).

5. Conclusion.

Since on specializing the parameters of generalized Gimel-function of several variables yield almost all special functions appearing in Applied Mathematics and Physical Sciences. Therefore the result presented in this study is of a general character and hence may encompass several cases of interest.

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