Fractional Calculus Operator Associated with Wright's Function

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ABSTRACT
The present paper aims at the study and derivation of Saigo generalized fractional integral operator involving product of multivariable Gimel-function , generalized polynomials and Wright function. On account of the most general nature of the operator, multivariable Aleph-function, generalized polynomials and Wright's function occuring in the main result, a large number of known and new results involving Riemann-Liouville, Erdelyi-Kober fractional differential operator, Bessel function, Mittag-leffler function follows as special cases of our main finding.

Keywords:Multivariable Gimel-function, Fox-Wright function, Saigo fractional integral operator, generalized multivariable polynomials, Appell function.

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1.Introduction.

We define a generalized transcendental function of several complex variables noted ℷ.

$$\gimel(z_1,\cdots,z_r)=\gimel^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c}z_1\\ \cdot\\ \cdot\\ \cdot\\ z_r\end{array}\right|$$

$$[(a_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}};[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},$$
$$[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{1,q_{i_2}};$$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(a_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{1,q_{i_3}};\cdots;\quad \cdots$$

$$[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{n_r+1,p_r}]:\ [(c^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]$$
$$[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{1,q_r}]:\ [(d^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]$$

$$\left.\begin{array}{l};\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})_{m^{(r)}+1,p^{(r)}_i}]\\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})_{n^{(r)}+1,q^{(r)}_i}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.5}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha^{(k)}_{2j}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha^{(k)}_{2ji_2}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta^{(k)}_{2ji2}s_k)]}$$

$$\frac{\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.6}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.7}$$

1) $[(c_j^{(1)};\gamma_j^{(1)}]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $n_2,\cdots,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2,\cdots,0\leqslant m_r,0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$

$0\leqslant n^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant p_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n^{(k)});(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$a_{kji_k}\in\mathbb{C};(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r)$.

$b_{kji_k}\in\mathbb{C};(j=1,\cdots,q_{i_k});(k=2,\cdots,r)$.

$d_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i=1,\cdots,R^{(k)}); (j=m^{(k)}+1,\cdots,q_{i^{(k)}}); (k=1,\cdots,r).$

$\gamma_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i=1,\cdots,R^{(k)}); (j=n^{(k)}+1,\cdots,p_{i^{(k)}}); (k=1,\cdots,r).$

The contour L_k is in the $s_k(k=1,\cdots,r)$- plane and run from $\sigma-i\infty$ to $\sigma+i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k\right)(j=1,\cdots,n_2), \Gamma^{A_{3j}}\left(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k\right)$ $(j=1,\cdots,n_3),\cdots,\Gamma^{A_{rj}}\left(1-a_{rj}+\sum_{i=1}^{r}\alpha_{rj}^{(i)}\right)(j=1,\cdots,n_r), \Gamma^{C_j^{(k)}}\left(1-c_j^{(k)}+\gamma_j^{(k)}s_k\right)(j=1,\cdots,n^{(k)})(k=1,\cdots,r)$to the right of the contour L_k and the poles of $\Gamma^{D_j^{(k)}}\left(d_j^{(k)}-\delta_j^{(k)}s_k\right)(j=1,\cdots,m^{(k)})(k=1,\cdots,r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2}A_i^{(k)}\pi$ where

$$A_i^{(k)} = \sum_{j=1}^{m^{(k)}} D_j^{(k)}\delta_j^{(k)} + \sum_{j=1}^{n^{(k)}} C_j^{(k)}\gamma_j^{(k)} - \tau_{i^{(k)}}\left(\sum_{j=m^{(k)}+1}^{q_i^{(k)}} D_{ji^{(k)}}^{(k)}\delta_{ji^{(k)}}^{(k)} + \sum_{j=n^{(k)}+1}^{p_i^{(k)}} C_{ji^{(k)}}^{(k)}\gamma_{ji^{(k)}}^{(k)}\right) +$$

$$-\tau_{i_2}\left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2}\alpha_{2ji_2}^{(k)} + \sum_{j=1}^{q_{i_2}} B_{2ji_2}\beta_{2ji_2}^{(k)}\right) - \cdots - \tau_{i_r}\left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r}\alpha_{rji_r}^{(k)} + \sum_{j=1}^{q_{i_r}} B_{rji_r}\beta_{rji_r}^{(k)}\right) \quad (1.8)$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,), max(\,|z_1|,\cdots,|z_r|\,) \rightarrow 0$

$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,), min(\,|z_1|,\cdots,|z_r|\,) \rightarrow \infty$ where $i=1,\cdots,r$:

$$\alpha_i = \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] \text{ and } \beta_i = \max_{1\leqslant j\leqslant n^{(i)}} Re\left[C_j^{(i)}\left(\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)\right]$$

Remark 1.
If $n_2=\cdots=n_{r-1}=p_{i_2}=q_{i_2}=\cdots=p_{i_{r-1}}=q_{i_{r-1}}=0$ and $A_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=A_{rji_r}=B_{rji_r}=1$ $A_{rj}=A_{rji_r}=B_{rji_r}=1$, then the multivariable Gimel-function reduces in the multivariable Aleph- function defined by Ayant [1].

Remark 2.
If $n_2=\cdots=n_r=p_{i_2}=q_{i_2}=\cdots=p_{i_r}=q_{i_r}=0$ and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2=\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [5].

Remark 3.
If$A_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=A_{rji_r}=B_{rji_r}=1$and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2=\cdots=R_r=R^{(1)}$ $=\cdots=R^{(r)}=1$, then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [4].

Remark 4.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H-function defined by Srivastava and panda [9,10].

In your investigation, we shall use the following notations.

$$\mathbb{A} =[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.9}$$

$$\mathbf{A} = [(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)},0,0;A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)},0,0;A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.10}$$

$$A = [(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \;;(1-a_i,\alpha_i;1)_{1,p'};\_ \tag{1.11}$$

$$\mathbb{B} = [\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})_{1,q_{i_{r-1}}}] \tag{1.12}$$

$$\mathbf{B} = [\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)},0,0;B_{rji_r})_{1,q_{i_r}}] \tag{1.13}$$

$$\mathrm{B} = [(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}];\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{m^{(r)}+1,q_i^{(r)}}];(1-b_j;\beta_j;1)_{1,q'+1};(0,1;1),(0,1;1) \tag{1.14}$$

$$U = 0,n_2;0,n_3;\cdots;0,n_{r-1};V = m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.15}$$

$$X = p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y = p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.16}$$

The generalized polynomials of multivariables defined by Srivastava [8], is given in the following manner :

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[y_1,\cdots,y_v]=\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\frac{(-N_1)_{\mathfrak{M}_1K_1}}{K_1!}\cdots\frac{(-N_v)_{\mathfrak{M}_vK_v}}{K_v!}A[N_1,K_1;\cdots;N_v,K_v]y_1^{K_1}\cdots y_v^{K_v} \tag{1.17}$$

where $\mathfrak{M}_1,\cdots,\mathfrak{M}_v$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_v,K_v]$ are arbitrary constants, real or complex.

We shall note

$$a_v =\frac{(-N_1)_{\mathfrak{M}_1K_1}}{K_1!}\cdots\frac{(-N_v)_{\mathfrak{M}_vK_v}}{K_v!}A[N_1,K_1;\cdots;N_v,K_v] \tag{1.18}$$

The Saigo fractional integral operator [6,7] is defined as :

$$I_{0,x}^{p,q,\gamma}f(x) = \left\{\begin{array}{l} \frac{x^{-p-q}}{\Gamma(p)}\int_0^x(x-t)^{p-1}F\left(p+q,-\gamma;p;1-\frac{t}{x}\right)f(t)\mathrm{d}t(Re(p)>0) \\ . \\ \frac{d^r}{dx^r}\ \mathrm{I}_{0,x}^{p+r,q-r,\gamma-r}f(x),(Re(p)\leqslant 0,0<Re(p)+r\leqslant 1,r=1,2,\cdots) \end{array}\right. \tag{1.19}$$

where F is the gauss hypergeometric function.

Saigo fractional integral operator contains as special cases, the Riemann-Liouville and Erdélyi-Kober operator of fractional integration of order $\alpha > 0$ [3].

$$I_{0,z}^{\alpha,-\alpha,-\alpha} f(z) = R^{\alpha} f(z) = \frac{z^{\alpha}}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(tz)\mathrm{d}t \tag{1.20}$$

$$z^{-\alpha-\gamma} I_{0,z}^{\alpha,-\alpha-\gamma,-\alpha} f(z) = I_1^{\gamma,\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} t^{\gamma} f(tz)\mathrm{d}t (\alpha > 0, \gamma \in \mathbb{R})$$

Let $\alpha, \alpha', \beta, \beta' \in \mathbb{R}$ and $\gamma > 0$, then Saigo generalized fractional integral operator [6] of a function $f(x)$ is defined by

$$I_{0,z}^{\alpha,\alpha',\beta,\beta',\gamma} f(z) = \frac{z^{-\alpha}}{\Gamma(\gamma)} \int_0^z (z-t)^{\gamma-1} t^{-\alpha'} F_3 \left[\alpha, \alpha', \beta, \beta'; \gamma'; 1-tz, 1-\frac{z}{t}\right] f(t)\mathrm{d}t (\gamma > 0) \tag{1.21}$$

where $f(z)$ is analytic in a simply connected region of $z-$plane. Principal value for $0 \leqslant arg(z-t) \leqslant 2\pi$ is denoted by $(z-t)^{\gamma-1}$.

The Appell hypergeometric function of third type denoted F_3 is defined by, see[8] :

$$F_3(\alpha, \alpha', \beta, \beta'; \gamma; z, t) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n (\beta)_m (\beta')_n}{(\gamma)_{m+n}} \frac{z^m t^n}{m!n!} \qquad |z| < 1, |t| < 1 \tag{1.22}$$

Also, the Fox-Wright function [11] is defined as

$$_{p'}\psi_{q'}(z) = {}_{p'}\psi_{q'} \left[\begin{array}{c|c} (\mathrm{e}_j, E_j)_{1,p'} & \\ . & \mathrm{z} \\ (\mathrm{f}_j, F_j)_{1,q'} & \end{array} \right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p'} \Gamma(e_j + E_j n)}{\prod_{j=1}^{q'} \Gamma(f_j + F_j n)} \frac{z^n}{n!} = \frac{1}{2i\pi} \int_{L_{+\infty}} \frac{\Gamma(s) \prod_{j=1}^{p'} \Gamma(e_j - E_j s)}{\prod_{j=1}^{q'} \Gamma(f_j - F_j s)} (-z)^{-s} \mathrm{d}s \tag{1.23}$$

Where $E_j (j = 1, \cdots, p')$ and $F_j (j = 1, \cdots, q')$ are real and positive numbers and verify

$$1 + \sum_{j=1}^{q'} F_j - \sum_{j=1}^{p'} E_j > 0$$

2. Required result.

Lemma. (see, Saigo and maeda [6])

Let $Re(\gamma) > 0, k > max[0, Re(\alpha + \alpha' + \beta - \gamma), Re(\alpha' - \beta')] - 1$ then

$$I_{0,z}^{\alpha,\alpha',\beta,\beta',\gamma}(z^k) = \frac{\Gamma(1+k)\Gamma(1+k-\alpha'+\beta')\Gamma(1+k-\alpha-\alpha'-\beta+\gamma)}{\Gamma(1+k+\beta')\gamma(1+k-\alpha'-\beta+\gamma)\gamma(1+k-\alpha-\alpha'+\gamma)} z^{k-\alpha-\alpha'+\gamma} \tag{2.1}$$

3. Main formula.

In this section, we establish a general formula.

Theorem.

$$I_{0,z}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma}(p-qt)^{\rho}\ {}_p\psi_{q'}(t^{\lambda}(p-qt)^{-\mu})\ S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\begin{pmatrix} t^{\mu_1}(p-qt)^{-\upsilon_1} \\ \cdot \\ \cdot \\ \cdot \\ t^{\mu_v}(p-qt)^{-\upsilon_v}\end{pmatrix} \gimel \begin{pmatrix} z_1 t^{\delta_1}(p-qt)^{-\eta_1} \\ \cdot \\ \cdot \\ \cdot \\ z_r t^{\delta_r}(p-qt)^{-\eta_r}\end{pmatrix}\right]$$

$$= \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} a_v t_1^{K_1}\cdots t_v^{K_v}\, p^{\rho-\sum_{i=1}^{v}\upsilon_i K_i} t^{\sigma+\sum_{i=1}^{v}\mu_i K_i-\alpha-\alpha'+\gamma}$$

$$\gimel_{X;p_{i_r}+4,q_{i_r}+4,\tau_{i_r}:R_r:Y;(m,n+1);(0;1)}^{U;0,n_r+4:V;(1,m);(1,0)}\left(\begin{array}{c|c} z_1 t^{\eta_1} & \mathbb{A};\ A_1, \mathbf{A} : A \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r t^{\eta_r} & \\ -t^{\lambda} & \mathbb{B};\ \mathbf{B}, B_1 : B \\ -t & \end{array}\right) \tag{3.1}$$

where

$$A_1 = (1+\rho-\sum_{i=1}^{v}\upsilon_i K_i; q_1,\cdots,q_r,\mu,1;1), (-\sigma-\sum_{i=1}^{v}\mu_i K_i; \eta_1,\cdots,\eta_r,\lambda,1;1),$$

$$(-\sigma-\sum_{i=1}^{v}\mu_i K_i+\alpha'-\beta'; \eta_1,\cdots,\eta_r,\lambda,1;1), (-\sigma-\sum_{i=1}^{v}\mu_i K_i+\alpha+\alpha'+\beta-\gamma; \eta_1,\cdots,\eta_r,\lambda,1;1) \tag{3.2}$$

$$B_1 = (1+\rho-\sum_{i=1}^{v}\upsilon_i K_i; q_1,\cdots,q_r,\mu,0;1), (-\sigma-\sum_{i=1}^{v}\mu_i K_i-\beta'; \eta_1,\cdots,\eta_r,\lambda,1;1),$$

$$(-\sigma-\sum_{i=1}^{v}\mu_i K_i+\alpha'+\beta'-\gamma; \eta_1,\cdots,\eta_r,\lambda,1;1), (-\sigma-\sum_{i=1}^{v}\mu_i K_i+\alpha+\alpha'-\gamma; \eta_1,\cdots,\eta_r,\lambda,1;1) \tag{3.3}$$

provided

$\alpha,\alpha',\beta,\beta',\gamma,\mu,\lambda,\sigma;,\rho\in\mathbb{C};\gamma,\mu_i,\upsilon_i>0(i=1,\cdots,v)$

$$Re(\sigma)+\sum_{i=1}^{r}\delta_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > max\{0,\alpha'-\beta',\alpha+\beta-\gamma\}-1.$$

$1+\sum_{j=1}^{q'}F_j-\sum_{j=1}^{p'}E_j>0$, $\left|\frac{q}{p}t\right|<1$ and $|arg(z_k)|<\frac{1}{2}A_i^{(k)}\pi$, $k=1,\cdots,r$ where $A_i^{(k)}$ is given in (1.4).

We obtain the Gimel-function of $(r+2)$-variables.

Proof

In order to prove (3.1), we first express the generalized polynomials in series form with the help of (1.17), the multivariable Gimel-function in terms of Mellin-Barnes type multiple integrals contour with the help of (1.1), the Fox-Wright ${}_p\psi_{q'}(.)$ in terms of contour integral with the help of equation (1.23) and then interchange the order of summations, integration and fractional integral operator, which is permissible under the stated conditions. Now using the lemma, we arrive at the desired result after an algebraic simplifications.

4. Particular case.

On account of the most general character of the multivariable Gimel-function, a class of multivariable polynomials and Fox-Wright function occurring in the main result, many special cases of the result can be derived but, for the sake of brevity, a case are recorded here.

Setting $p' = 1 = q', \alpha_= a_1 = 1, b_1 = b', \beta_1 = a'$ in (3.1), we get

Corollary.

$$I_{0,z}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma}(p-qt)^{\rho}\,E_{A',B'}\left(t^{\lambda}(p-qt)^{-\mu_1}\right)\,S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\begin{pmatrix} t^{\mu_1}(p-qt)^{-\upsilon_1} \\ \cdot \\ \cdot \\ \cdot \\ t^{\mu_v}(p-qt)^{-\upsilon_v}\end{pmatrix}\beth\begin{pmatrix} z_1 t^{\delta_1}(p-qt)^{-\eta_1} \\ \cdot \\ \cdot \\ \cdot \\ z_r t^{\delta_r}(p-qt)^{-\eta_r}\end{pmatrix}\right]$$

$$= \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} a_v t_1^{K_1}\cdots t_v^{K_v}\, p^{\rho-\sum_{i=1}^{v}\upsilon_i K_i} t^{\sigma+\sum_{i=1}^{v}\mu_i K_i-\alpha-\alpha'+\gamma}$$

$$\beth_{X;p_{i_r}+4,q_{i_r}+4,\tau_{i_r}:R_r:Y;(1,2);(0;1)}^{U;0,n_r+4:V;(1,1);(1,0)}\left(\begin{array}{c|c} z_1 t^{\eta_1} & \\ \cdot & \mathbb{A};\, \mathrm{A}_2, \mathbf{A} : A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r t^{\eta_r} & \cdot \\ \text{-}t^{\lambda} & \mathbb{B};\, \mathbf{B}, \mathrm{B}_2 : B \\ \text{-t} & \end{array}\right) \tag{4.1}$$

where

$$A_2 = (1+\rho-\sum_{i=1}^{v}\upsilon_i K_i; q_1,\cdots,q_r,\mu,1;1), (-\sigma-\sum_{i=1}^{v}\mu_i K_i;\eta_1,\cdots,\eta_r,\lambda,1;1),$$

$$(-\sigma-\sum_{i=1}^{v}\mu_i K_i+\alpha'-\beta';\eta_1,\cdots,\eta_r,\lambda,1;1), (-\sigma-\sum_{i=1}^{v}\mu_i K_i+\alpha+\alpha'+\beta-\gamma;\eta_1,\cdots,\eta_r,\lambda,1;1) \tag{4.2}$$

$$B_2 = (1+\rho-\sum_{i=1}^{v}\upsilon_i K_i; q_1,\cdots,q_r,\mu,0;1), (-\sigma-\sum_{i=1}^{v}\mu_i K_i-\beta';\eta_1,\cdots,\eta_r,\lambda,1;1),$$

$$(-\sigma-\sum_{i=1}^{v}\mu_i K_i+\alpha'+\beta'-\gamma;\eta_1,\cdots,\eta_r,\lambda,1;1), (-\sigma-\sum_{i=1}^{v}\mu_i K_i+\alpha+\alpha'-\gamma;\eta_1,\cdots,\eta_r,\lambda,1;1) \tag{4.3}$$

$$A = [(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}], [\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}], [\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}]\ ;(0,1;1);\_ \tag{4.4}$$

and

$$\mathrm{B} = [(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}], [\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}];\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}], [\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{m^{(r)}+1,q_i^{(r)}}];(1-B';A';1);(0,1;1);(0,1;1) \tag{4.5}$$

provided that

$\alpha,\alpha',\beta,\beta',\gamma,\mu,\lambda,\sigma;,\rho\in\mathbb{C};\gamma,\mu_i,\upsilon_i>0(i=1,\cdots,v)$

$$Re(\sigma)+\sum_{i=1}^{r}\delta_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > max\{0,\alpha'-\beta',\alpha+\beta-\gamma\}-1.$$

$\left|\frac{q}{p}t\right| < 1$ and $|arg(z_k)| < \frac{1}{2}A_i^{(k)}\pi$, $k=1,\cdots,r$ where $A_i^{(k)}$ is given in (1.4).

5. Conclusion.

The results derived in the present paper are quite general in nature so a large number of known and new results involving Riemann-Liouville, Erdélyi- Kober Fractional differential operators,Bessel function, Mittag-leffler function etc. can be obtained from it.

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