Integration of Certain Products Associated with the Multivariable Gimel-Function

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ABSTRACT
In this paper, we evaluate two infinite integrals in terms of the multivariable Gimel-function defined here. The scope of a further generalization of these results, with the aid of the Mellin inversion theorem, is also discussed.

KEYWORDS : Multivariable Gimel-function, multiple integral contours, Whittaker function, Mellin inversion theorem.

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1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C}, \mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

We define a generalized transcendental function of several complex variables noted $\gimel$.

$$\gimel(z_1, \cdots, z_r) = \gimel^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}} \left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array} \right|$$

$$[(a_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}};[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},$$
$$[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{1,q_{i_2}};$$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(a_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{1,q_{i_3}};\cdots; \quad \cdots$$

$$[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{n_r+1,p_r}]: \ [(c^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]$$
$$[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{1,q_r}]: \ [(d^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]$$

$$\left. \begin{array}{l} ;\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})_{m^{(r)}+1,p^{(r)}_i}] \\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})_{n^{(r)}+1,q^{(r)}_i}] \end{array} \right)$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1,\cdots,s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k} \, \mathrm{d}s_1 \cdots \mathrm{d}s_r \tag{1.5}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha^{(k)}_{2j}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha^{(k)}_{2ji_2}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta^{(k)}_{2ji2}s_k)]}$$

$$\frac{\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.6}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.7}$$

1) $[(c_j^{(1)};\gamma_j^{(1)}]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $n_2,\cdots,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2,\cdots,0\leqslant m_r,0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$

$0\leqslant n^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant p_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n^{(k)});(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$a_{kji_k}\in\mathbb{C};(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r)$.

$b_{kji_k} \in \mathbb{C}; (j = 1, \cdots, q_{i_k}); (k = 2, \cdots, r)$.

$d^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = m^{(k)} + 1, \cdots, q_{i^{(k)}}); (k = 1, \cdots, r)$.

$\gamma^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r)$.

The contour L_k is in the $s_k(k = 1, \cdots, r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1 - a_{2j} + \sum_{k=1}^{2} \alpha^{(k)}_{2j} s_k\right) (j = 1, \cdots, n_2), \Gamma^{A_{3j}}\left(1 - a_{3j} + \sum_{k=1}^{3} \alpha^{(k)}_{3j} s_k\right)$ $(j = 1, \cdots, n_3), \cdots, \Gamma^{A_{rj}}\left(1 - a_{rj} + \sum_{i=1}^{r} \alpha^{(i)}_{rj}\right) (j = 1, \cdots, n_r), \Gamma^{C^{(k)}_j}\left(1 - c^{(k)}_j + \gamma^{(k)}_j s_k\right) (j = 1, \cdots, n^{(k)})(k = 1, \cdots, r)$to the right of the contour L_k and the poles of $\Gamma^{D^{(k)}_j}\left(d^{(k)}_j - \delta^{(k)}_j s_k\right) (j = 1, \cdots, m^{(k)})(k = 1, \cdots, r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2} A^{(k)}_i \pi$ where

$$A^{(k)}_i = \sum_{j=1}^{m^{(k)}} D^{(k)}_j \delta^{(k)}_j + \sum_{j=1}^{n^{(k)}} C^{(k)}_j \gamma^{(k)}_j - \tau_{i^{(k)}} \left(\sum_{j=m^{(k)}+1}^{q^{(k)}_i} D^{(k)}_{ji^{(k)}} \delta^{(k)}_{ji^{(k)}} + \sum_{j=n^{(k)}+1}^{p^{(k)}_i} C^{(k)}_{ji^{(k)}} \gamma^{(k)}_{ji^{(k)}} \right) +$$

$$-\tau_{i_2} \left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2} \alpha^{(k)}_{2ji_2} + \sum_{j=1}^{q_{i_2}} B_{2ji_2} \beta^{(k)}_{2ji_2} \right) - \cdots - \tau_{i_r} \left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r} \alpha^{(k)}_{rji_r} + \sum_{j=1}^{q_{i_r}} B_{rji_r} \beta^{(k)}_{rji_r} \right) \tag{1.8}$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(\,|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}\,), max(\,|z_1|, \cdots, |z_r|\,) \rightarrow 0$

$\aleph(z_1, \cdots, z_r) = 0(\,|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}\,), min(\,|z_1|, \cdots, |z_r|\,) \rightarrow \infty$ where $i = 1, \cdots, r$:

$$\alpha_i = \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D^{(i)}_j \left(\frac{d^{(i)}_j}{\delta^{(i)}_j}\right)\right] \text{ and } \beta_i = \max_{1 \leqslant j \leqslant n^{(i)}} Re\left[C^{(i)}_j \left(\frac{c^{(i)}_j - 1}{\gamma^{(i)}_j}\right)\right]$$

Remark 1.
If $n_2 = \cdots = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$ $A_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the multivariable Gimel-function reduces in the multivariable Aleph- function defined by Ayant [1].

Remark 2.
If $n_2 = \cdots = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [8].

Remark 3.
If$A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [7].

Remark 4.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H-function defined by Srivastava and panda [9 ,10].

In your investigation, we shall use the following notations.

$$\mathbb{A} =[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \quad (1.9)$$

$$\mathbf{A} = [(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \quad (1.10)$$

$$A = [(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \quad (1.11)$$

$$\mathbb{B} = [\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})_{1,q_{i_{r-1}}}] \quad (1.12)$$

$$\mathbf{B} = [\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{1,q_{i_r}}] \quad (1.13)$$

$$\mathrm{B} = [(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}];\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \quad (1.14)$$

$$U = 0,n_2;0,n_3;\cdots;0,n_{r-1};V = m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \quad (1.15)$$

$$X = p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y = p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \quad (1.16)$$

2. Required integral.

We evaluate the following integral with a modified from of the formula ([4], p. 410, (42)).

Lemma 1.

$$\int_0^\infty x^{\rho-1}W_{k,\mu}(x)W_{\lambda,\upsilon}(x)\mathrm{d}x = \frac{\Gamma(-2\upsilon)}{\Gamma\left(\frac{1}{2}-\lambda-\upsilon\right)}\sum_{t=0}^{\infty}\frac{\left(\frac{1}{2}-\lambda+\upsilon\right)_t\Gamma(1+\mu+\upsilon+\rho+t)\Gamma(1-\mu+\upsilon+\rho+t)}{t!(1+2\upsilon)_t\Gamma\left(\frac{3}{2}-k+\upsilon+\rho+t\right)}$$

$$+\frac{\Gamma(2\upsilon)}{\Gamma\left(\frac{1}{2}-\lambda+\upsilon\right)}\sum_{t=0}^{\infty}\frac{\left(\frac{1}{2}-\lambda-\upsilon\right)_t\Gamma(1+\mu+\upsilon+\rho+t)\Gamma(1-\mu-\upsilon+\rho+t)}{t!(1-2\upsilon)_t\Gamma\left(\frac{3}{2}-k-\upsilon+\rho+t\right)} \quad (2.1)$$

provided $|Re(\mu)|+|Re(\upsilon)|<\rho+1$.

3. Main results.

We shall use the symbol $\sum_{\sigma,-\sigma}$ in this document, this last indicates that to the following expression it as a similar expression, with σ replaced by $-\sigma$, is to be added.

Theorem 1.

$$\int_0^\infty t^{\rho-1}W_{k,\mu}(t)W_{\upsilon,\sigma}(t)\gimel(z_1t^{a_1},\cdots,z_rt^{a_r})\mathrm{d}t=\sum_{\sigma,-\sigma}\frac{\Gamma(-2\sigma)}{\Gamma\left(\frac{1}{2}-\upsilon-\sigma\right)}\sum_{u=0}^{\infty}\frac{\left(\frac{1}{2}-\upsilon+\sigma\right)_u}{(1+2\sigma)_u}$$

$$\gimel^{U;0,n_r+2:V}_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};\ (\pm\mu-\sigma-\rho-u;a_1,\cdots,a_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\ \mathbf{B},(k-\sigma-\rho-u-\frac{1}{2};a_1,\cdots,a_r;1):B \end{array}\right) \tag{3.1}$$

provided

$$a_i>0(i=1,\cdots,r),\ Re(\rho)+\sum_{i=1}^{r}a_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>|Re(\mu)|+|Re(\sigma)|-1$$

$|arg(z_it^{a_i})|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Proof
Expressing the multivariable Gimel-function in the integrand in Mellin-Barnes multiple integrals contour with the help of (1.1), and interchanging the order of integrations (which is permissible under the conditions mentioned in (2.1)), we get

$$\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\left[\int_0^\infty x^{\sigma+\sum_{i=1}^r a_is_i-1}W_{k,\mu}(x)W_{\upsilon,\sigma}(x)\mathrm{d}x\right]\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.2}$$

Evaluating the inner-integral with the help of lemma 1 and interchanging the order of integration and summations which is justified under the conditions mentioned above, we get

$$\sum_{\sigma,-\sigma}\frac{\Gamma(-2\sigma)}{\Gamma\left(\frac{1}{2}-\upsilon-\sigma\right)}\sum_{u=0}^{\infty}\frac{\left(\frac{1}{2}-\upsilon+\sigma\right)_u}{(1+2\sigma)_u}\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}$$

$$\frac{\Gamma(\mp\mu+\upsilon+\rho+\sum_{i=1}^r a_is_i+u)}{\Gamma\left(\frac{3}{2}-k+\upsilon+\rho+\sum_{i=1}^r a_is_i+u\right)}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.3}$$

Interpreting the resulting expression with the help of (1.1), we obtain the desired result (3.1).

Theorem 2.

$$\int_0^\infty t^{2\rho-1}K_{2\mu}(\alpha t)K_{2\upsilon}(\beta t)\gimel(z_1t^{2a_1},\cdots,z_rt^{2a_r})\mathrm{d}t=\frac{\sqrt{\pi}\beta^{2\upsilon}}{4\alpha^{2(\upsilon+\rho)}}\sum_{u=0}^{\infty}\frac{\left(1-\left(\frac{\beta}{\alpha}\right)^2\right)^u}{2^uu!}\gimel^{U;0,n_r+4:V}_{X;p_{i_r}+4,q_{i_r}+2,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} \mathrm{z}_1\alpha^{-2a_1} & \mathbb{A};\ (1-\rho\pm\mu+\upsilon;a_1,\cdots,a_r;1),(1-\rho\pm\mu-\upsilon-u;a_1,\cdots,a_r;1),,\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r\alpha^{-2a_r} & \mathbb{B};\ \mathbf{B},\left(1-\rho-\frac{u}{2};a_1,\cdots,a_r;1\right),\left(\frac{1-u}{2}-\rho;a_1,\cdots,a_r;1\right):B \end{array}\right) \tag{3.4}$$

provided

$$Re(\alpha+\beta),a_i>0(i=1,\cdots,r),\ Re(\rho)+\sum_{i=1}^{r}a_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>|Re(\mu)|+|Re(\sigma)|.$$

$\left|arg(z_it^{2a_i})\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

To prove the theorem 2, we use the similar manner with the help of formula ([3], Vol. I, p. 334, Eq. (47)).

4. Special cases.

In this section, we shall see several particular cases. Taking $k = \upsilon = 0$ in (3.1) and using the formula ([4], Vol. II, p. 432),

$$W_{0,\mu}(2z) = \left(\frac{2z}{\pi}\right)^{\frac{1}{2}} K_\mu(z) \tag{4.1}$$

we obtain the following integral

Corollary 1.

$$\int_0^\infty t^{2\rho-1} K_{2\mu}(\alpha t) K_{2\upsilon}(\alpha t) \gimel(z_1 t^{a_1}, \cdots, z_r t^{a_r}) \mathrm{d}t = \frac{\pi}{(2\alpha)^{2\rho}} \sum_{\upsilon,-\upsilon} \frac{\Gamma(-4\upsilon)}{\Gamma\left(\frac{1}{2}-2\upsilon\right)} \sum_{u=0}^\infty \frac{\left(\frac{1}{2}+2\upsilon\right)_u}{(1+4\upsilon)_u}$$

$$\gimel^{U;0,n_r+2:V}_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} \frac{z_1}{(2\alpha)^{a_1}} & \mathbb{A}; (1 \pm 2\mu - 2\upsilon - 2\rho - u; a_1, \cdots, a_r; 1), \mathbf{A} : A \\ \cdot & \cdot \\ \cdot & \cdot \\ \frac{z_r}{(2\alpha)^{a_r}} & \mathbb{B}; \mathbf{B}, (\frac{1}{2} - 2\upsilon - 2\rho - u; a_1, \cdots, a_r; 1) : B \end{array}\right) \tag{4.2}$$

provided

$$Re(\alpha), a_i > 0 (i = 1, \cdots, r), Re(\rho) + \frac{1}{2}\sum_{i=1}^r a_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > |Re(\mu)| + |Re(\sigma)|$$

$|arg(z_i t^{a_i})| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Taking $\alpha = \beta$, the infinite series on the right-hand side of equation (3.4) reduce to its first term given by $t = 0$, and we have

Corollary 2.

$$\int_0^\infty t^{2\rho-1} K_{2\mu}(\alpha t) K_{2\upsilon}(\alpha t) \gimel(z_1 t^{2a_1}, \cdots, z_r t^{2a_r}) \mathrm{d}t = \frac{\sqrt{\pi}}{4\alpha^{2\rho}}$$

$$\gimel^{U;0,n_r+4:V}_{X;p_{i_r}+4,q_{i_r}+2,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} \frac{z_1}{\alpha^{2a_1}} & \mathbb{A}; (1-\rho \pm \mu + \upsilon; a_1, \cdots, a_r; 1), (1-\rho \pm \mu - \upsilon; a_1, \cdots, a_r; 1), \mathbf{A} : A \\ \cdot & \cdot \\ \cdot & \cdot \\ \frac{z_r}{\alpha^{2a_r}} & \mathbb{B}; \mathbf{B}, (1-\rho; a_1, \cdots, a_r; 1), \left(\frac{1}{2} - \rho; a_1, \cdots, a_r; 1\right) : B \end{array}\right) \tag{4.3}$$

provided

$$Re(\alpha), a_i > 0 (i = 1, \cdots, r), Re(\rho) + \sum_{i=1}^r a_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > |Re(\mu)| + |Re(\sigma)|.$$

$\left|arg(z_i t^{2a_i})\right| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

The first member of equation (4.2) with taking $a_i \to 2a_i$ is the same as that of the above equation ; thus, by equating their right-hand sides, we obtain the following relation :

Corollary 3.

$$\daleth_{X;p_{i_r}+4,q_{i_r}+2,\tau_{i_r}:R_r:Y}^{U;0,n_r+4:V}\left(\begin{array}{c|c} \frac{z_1}{\alpha^{2a_1}} & \mathbb{A};\,(1-\rho\pm\mu+\upsilon;a_1,\cdots,a_r;1),(1-\rho\pm\mu-\upsilon;a_1,\cdots,a_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \frac{z_r}{\alpha^{2a_r}} & \mathbb{B};\,\mathbf{B},(1-\rho;a_1,\cdots,a_r;1)\,,\left(\frac{1}{2}-\rho;a_1,\cdots,a_r;1\right):B \end{array}\right)$$

$$=\sqrt{2}\sum_{\upsilon,-\upsilon}\frac{\Gamma(-4\upsilon)}{\Gamma\left(\frac{1}{2}-2\upsilon\right)}\sum_{u=0}^{\infty}\frac{2^{2\upsilon+u}\left(\frac{1}{2}+2\upsilon\right)_u}{(1+4\upsilon)_u}$$

$$\daleth_{X;p_{i_r}+4,q_{i_r}+2,\tau_{i_r}:R_r:Y}^{U;0,n_r+4:V}\left(\begin{array}{c|c} z_1 & \mathbb{A};\,(1-\rho\pm\mu-\upsilon-\frac{u}{2};a_1,\cdots,a_r;1),(\frac{1-u}{2}-\rho\pm\mu-\upsilon;a_1,\cdots,a_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\,\mathbf{B},(\frac{1-2u}{4}-\mu-\rho;a_1,\cdots,a_r;1)\,,(\frac{3-2u}{4}-\upsilon-\rho;a_1,\cdots,a_r;1):B \end{array}\right) \tag{4.4}$$

under the same existence conditions that theorem .

If we take $\upsilon=\sigma+\frac{1}{2}$ in theorem 1 and use the relation ([4], Vol. II, p. 432),

$$W_{\sigma+\frac{1}{2},\sigma}(z)=z^{\sigma+\frac{1}{2}}e^{-\frac{z}{2}} \tag{4.5}$$

we have the following integral

Corollary 4.

$$\int_0^{\infty}t^{\rho-1}e^{-\frac{t}{2}}W_{k,\mu}(t)\daleth(z_1t^{a_1},\cdots,z_rt^{a_r})\mathrm{d}t=$$

$$\daleth_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}^{U;0,n_r+2:V}\left(\begin{array}{c|c} z_1 & \mathbb{A};\,\left(\frac{1}{2}\pm\mu-\rho;a_1,\cdots,a_r;1\right),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\,\mathbf{B},(k-\rho-t;a_1,\cdots,a_r;1):B \end{array}\right) \tag{4.6}$$

provided

$$a_i>0(i=1,\cdots,r),\,Re(\rho)+\sum_{i=1}^{r}a_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>|Re(\mu)|-\frac{1}{2}$$

$|arg(z_it^{a_i})|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

In the above integral take $k=m+\frac{\alpha+1}{2},\mu=\frac{\alpha}{2}$, and use the formula ([4], Vol. II, p. 432)

$$W_{\mu+m+\frac{1}{2},m}(z)=(-)^m m!z^{\mu+\frac{1}{2}}L_m^{(2\mu)}(z),0\leqslant m \tag{4.7}$$

where $L_m^{(\alpha)}(x)$ is the Laguerre polynomial of order α and degree m, we get

Corollary 5.

$$\int_0^{\infty}t^{\rho+\frac{\alpha}{2}-1}e^{-t}L_m^{(\alpha)}(t)\daleth(z_1t^{a_1},\cdots,z_rt^{a_r})\mathrm{d}t=\frac{(-)^m}{m!}$$

$$\daleth_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}^{U;0,n_r+2:V}\left(\begin{array}{c|c} z_1 & \mathbb{A};\,\left(1-\rho\pm\frac{\alpha}{2};a_1,\cdots,a_r;1\right),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\,\mathbf{B},\,\left(1-\rho+m+\frac{\alpha}{2};a_1,\cdots,a_r;1\right):B \end{array}\right) \tag{4.8}$$

provided

$$a_i > 0(i = 1, \cdots, r), Re\left(\rho + \frac{\alpha}{2}\right) + \sum_{i=1}^{r} a_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0$$

$|arg(z_i t^{a_i})| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

5. Generalization.

If we let

$$A(t) = \frac{1}{2\pi\omega}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\prod_{j=1}^{q}\Gamma(v_j + m_j s)}{\prod_{j=1}^{n}\Gamma(w_j - n_j s)\prod_{j=1}^{p}\Gamma(u_j - l_j s)}t^{-s}\mathrm{d}s \tag{5.1}$$

where $\sigma > 0$, the $l's, m's, n's$ are are positive numbers and verify the relation

$$\sum_{j=1}^{p} l_j - \sum_{j=1}^{q} m_j - \sum_{j=1}^{m} n_j \leqslant 0 \tag{5.2}$$

The function defined above can be expressed in terms of the H-function of Fox ([5], p. 408), we have ([11], p. 265, Eq. (1.1)),

$$A(t) = H_{p,q+m}^{q,0}\left[\ \mathrm{t}\ \middle|\ \begin{matrix}(\mathrm{u}_j, l_j)_{1,p}\\ .\\ (\mathrm{v}_j, m_j)_{1,q}, (1-w_j, n_j)_{1,m}\end{matrix}\right] \tag{5.3}$$

provided $|arg(t)| < \frac{1}{2}\left(\sum_{j=1}^{q} m_j - \sum_{j=1}^{m} n_j - \sum_{j=1}^{p} l_j\right) > 0$

Use the Mellin inversion theorem, (5.1) evidently yields

Lemma 2.

$$\int_0^\infty t^{s-1}A(t)\mathrm{d}t = \frac{\prod_{j=1}^{q}\Gamma(v_j + m_i s)}{\prod_{j=1}^{m}\Gamma(w_j - n_i s)\prod_{j=1}^{p}\Gamma(u_j + l_i s)} \tag{5.4}$$

provided that $Re(s) > -\min_{1 \leqslant j \leqslant q} Re\left(\frac{v_j}{m_j}\right)$, and the integral $\int_0^\infty t^{s-1}|A(t)|\mathrm{d}t$ is bounded for some $s > 0$ where $\sigma > s$

We obtain the general integral

Theorem 3.

$$\int_0^\infty t^{\rho-1}A(t)\daleth(z_1 t^{a_1}, \cdots, z_r t^{a_r})\mathrm{d}t = \daleth_{X;p_{i_r}+q+m, q_{i_r}+p, \tau_{i_r}:R_r:Y}^{U;0,n_r+q:V}$$

$$\left(\begin{matrix}\mathrm{z}_1\\ .\\ .\\ \mathrm{z}_r\end{matrix}\ \middle|\ \begin{matrix}\mathbb{A}; (1 - v_j - \rho m_j; m_j a_1, \cdots, m_j a_r; 1)_{1,n}, \mathbf{A}, (w_j - \rho n_j; n_j a_1, \cdots, n_j a_r; 1)_{1,m} : A\\ .\\ .\\ \mathbb{B}; (1 - u_j - \rho l_j; l_j a_1, \cdots, l_j a_r; 1)_{1,p}, \mathbf{B} : B\end{matrix}\right) \tag{5.5}$$

provided

$$a_i > 0(i = 1, \cdots, r), Re(\rho) + \sum_{i=1}^{r} a_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -\min_{1 \leqslant j \leqslant q} Re\left(\frac{v_j}{m_j}\right)$$

$|arg(z_i t^{a_i})| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4) and $\sum_{j=1}^{q} m_j - \sum_{j=1}^{m} n_j - \sum_{j=1}^{p} l_j > 0$.

Proof

Expressing the multivariable Gimel-function in the integrand in Mellin-Barnes multiple integrals contour with the help of (1.1), and interchanging the order of integrations (which is permissible under the conditions mentioned in (5.5)), we get

$$\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\left[\int_0^\infty t^{\rho+\sum_{i=1}^r a_i s_i-1}A(t)\mathrm{d}t\right]\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{5.6}$$

Evaluating the inner-integral with the help of lemma 2, we get

$$\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\frac{\prod_{j=1}^{q}\Gamma(v_j+m_i(\rho+\sum_{i=1}^r a_i s_i))}{\prod_{j=1}^{m}\Gamma(w_j-n_i(\rho+\sum_{i=1}^r a_i s_i))\prod_{j=1}^{p}\Gamma(u_j+l_i(\rho+\sum_{i=1}^r a_i s_i))}$$

$$\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{5.7}$$

Interpreting the resulting expression with the help of (1.1), we obtain the desired result (5.5).

Remark :

Panda [6] have obtainded the same integrals about the multivariable H-function defined by Srivastava and Panda [9,10].

4. Conclusion.

A large number of other integrals involving Whittaker functions, Bessel associated functions, Laguerre polynomials can also be obtained from on account of the most general nature of the multivariable Gimel-function. The last function can be suitably specialized to a remarkably wide variety of special functions (or product of several such special functions) which are expressible in terms of E, F, G, H, I, Aleph-functions of one or more variables.

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