Compatible Maps and Some Common Fixed Point Results in G-Metric Space

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Abstract

In this paper we study compatible maps in G-Metric space and proved common fixed point theorems for pair of Compatible maps which satisfies contractive condition involving maximum function.

Keywords: G-Metric Space,G-Cauchy sequence,G-convergent Sequence,Common fixed point.

2000 Mathematics Subject Classification: 47H10, 54H25

1 Introduction

In 1976, G.Jungck [1] proved a common fixed point theorem for commuting mappings,which generalizes the Banach Contraction principle.Sesa [2] introduced a concept of weakly commuting mappings and proved some fixed point theorems in complete metric space.Commuting maps are weakly commuting.Jungck's [1] common fixed point theorem has been generalized and modified by many authors [3, 4, 5, 6, 7].In 1986 G.Jungck [5] defined the concept of compatibility and proved some common fixed point results.

In 2006 Mustafa and Sims [9] introduced the concept of G-Metric space.In 2012 Manoj Kumar [8] defined the concept of compatible maps in G-Metric space and proved some results of common fixed points of pair of compatible maps.Recently Latpate V.V. and Dolhare U.P. [10] proved one result of common fixed point theorem for pair of compatible mappings in G-Metric space.

2 Preliminaries

Definition 2.1. *Let X be a non empty set and* $G : X^3 \to R^+$ *which satisfies the following conditions*

1. $G(a, b, c) = 0\ if a = b = c$ *i.e. every* a, b, c *in* X *coincides.*
2. $G(a, a, b) > 0\ for\ every\ a, b, c\ \epsilon X$ *s.t.* $a \neq b$
3. $G(a, a, b) \leq G(a, b, c),\ \forall a, b, c\ \epsilon\ \ X$ *s.t.* $c \neq b$
4. $G(a, b, c) = G(b, a, c) = G(c, b, a) = ...$
   (symmetrical in all three variables)
5. $G(a, b, c) \leq G(a, x, x) + G(x, b, c)$, *for all a,b,c,x in X*
   (rectangle inequality)

   Then the function G is said to be generalized metric or simply G-metric on X and the pair (X,G) is said to be G-metric space.

Example 2.2. *Let* $G : X^3 \to R^+$ *s.t.* $G(a, b, c) =$ *perimeter of the triangle with vertices at a,b,c in* R^2 *, also by taking p in the interior of the triangle then rectangle inequality is satisfied and the function G is a G-metric on X.*

Remark 2.3. *G-metric space is the generalization of the ordinary metric space that is every G-metric space is (X,G) defines ordinary metric space* (X, d_G) *by*

$$d_G(a, b) = G(a, b, b) + G(a, a, b)$$

Example 2.4. *Let (X,d) be the usual metric space . Then the function* $G : X^3 \to R^+$ *defined by*

$$G(a, b, c) = max.\{d(a, b), d(b, c), d(c, a)\}$$

for all a,b,c ϵ *X is a G-metric space.*

Definition 2.5. *A G-metric space (X,G) is said to be symmetric if G(a,b,b)=G(a,a,b) for all a,b*ϵ*X and if G(a,b,b)*$\neq$ *G(a,a,b) then G is said to be non symmetric G-metric space.*

Example 2.6. *Let X=*$\{x, y\}$ *and* $G : X^3 \to R^+$ *defined by G(x,x,x)=G(y,y,y)=0, G(x,x,y)=1,G(x,y,y)=2 and extend G to all of* X^3 *by symmetry in the variables.Then X is a G-metric space but It is non symmetric. since G(x,x,y)* $\neq$ *G(x,y,y)*

Definition 2.7. *Let (X,G) be a G-metric space, Let* $\{ a_n \}$ *be a sequence of elements in X .The sequence* $\{ a_n \}$ *is said to be G-convergent to a if*

$$lim_{m,n\to\infty}G(a, a_n, a_m) = 0$$

i.e for every $\epsilon > 0$ *there is N s.t.* $G(a, a_n, a_m) < \epsilon$ *for all m,n* $\geq N$ *It is denoted as* $a_n \to a$ *or* $lim_{n\to\infty}a_n = a$

Proposition 2.8. *If (X,G) be a G-metric space.Then the following are equivalent*

1. $\{a_n\}$ *is G-convergent to a.*
2. $G(a_n, a_n, a) \to 0$ *as* $n \to \infty$
3. $G(a_n, a, a) \to 0$ *as* $n \to \infty$
4. $G(a_m, a_n, a) \to 0$ *as* $m, n \to \infty$

Definition 2.9. *Let (X,G) be a G-metric space a sequence* $\{a_n\}$ *is called G-Cauchy if , for each* $\epsilon > 0$ *there is an N* ϵI^+ *(set of positive integers) s.t.*

$$G(a_n, a_m, a_l) < \epsilon \; for \; all \; n, m, l \geq N$$

Proposition 2.10. *Let (X,G) be a G-metric space then the function G(a,b,c) is jointly continuous in all three of its variables.*

Proposition 2.11. *Let (X,G) be a G-metric space. Then, for any a,b,c,x in X it gives that*

1. $if\ G(a,b,c) = 0\ then\ a = b = c$
2. $G(a,b,c) \leq G(a,a,b) + G(a,a,c)$
3. $G(a,b,b) \leq 2G(b,a,a)$
4. $G(a,b,c) \leq G(a,x,c) + G(x,b,c)$
5. $G(a,b,c) \leq \frac{2}{3}(G(a,x,x) + G(b,x,x) + G(c,x,x))$

Definition 2.12. *[5] Let S and T be two self maps on a metric space (X,d). The mappings S and T are said to be compatible if*

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$$

, whenever $\{x_n\}$ *is a sequence in X such that*
$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z\ for\ some\ z\epsilon X$

Definition 2.13. *[8] Let S and T be two self mappings on a G-metric space (X,G). Then mappings S and T are said to be compatible if* $\lim_{n\to\infty} G(STx_n, STx_n, TSx_n) = 0$ *, whenever* $\{x_n\}$ *is a sequence in X s.t.* $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z\ for\ some\ z\epsilon X$ *for some z in X.*

Example 2.14. *Let X=[-1,1] and* $G : X^3 \to R^+$ *be defined as follows*

$$G(a,b,c) = |a-b| + |b-c| + |c-a|$$

for all a,b,c ϵX *. Then (X,G) be a G-metric space. Let us define fa=a and ga=*$\frac{a}{4}$ *Let* $\{a_n\}$ *be the sequence, s.t.* $a_n = \frac{1}{n}$ *and n is a natural number. It is easy to see that the mappings f and g are compatible as* $\lim_{n\to\infty} G(fga_n, gfa_n, gfa_n) = 0$ *here* $a_n = \frac{1}{n}$ *s.t.* $\lim_{n\to\infty} fa_n = \lim_{n\to\infty} ga_n = 0$ *for* $0\epsilon X$

Now we see some preliminary results of common fixed point theorem as follows. Manoj Kumar Generalized following theorem.Which is stated as

Theorem 2.15. *[8] Let(X,G) be complete G-metric space.Let S and T be self mappings on X satisfying following conditions.*

1. *$S(X) \subseteq T(X)$,*
2. *S or T is continuous ,*
3. *$G(Sa, Sb, Sc) \leq \beta G(Ta, Tb, Tc)$ for every a,b,c in X and $0 \leq \beta < 1$ .And if S and T are Compatible then S and T have Unique common fixed points in X.*

Proof. Let us take a_0 be an arbitrary element of X.We define a sequence s.t. for any point a_1 in X, define $Sa_0 = Ta_1$, $Sa_1 = Ta_2$, $Sa_2 = Ta_3$,....,In general for $a_{n+1} \in X$ s.t. $b_n = Sa_n = Ta_{n+1}$ for $n = 0, 1, 2, 3....$ from (3) we get

$$\begin{aligned} G(Sa_n, Sa_{n+1}, Sa_{n+1}) &\leq \beta G(Ta_n, Ta_{n+1}, Ta_{n+1}) \\ &= \beta G(Sa_{n-1}, Sa_n, Sa_n) \end{aligned}$$

By continuing same procedure, we get

$$G(Sa_n, Sa_{n+1}, Sa_{n+1}) \leq \beta^n G(Sa_0, Sa_1, Sa_1) \tag{2.1}$$

$\therefore$ for all $n, m \in N$, $m > n$, by using rectangle inequality, we get

$$\begin{aligned} G(b_n, b_m, b_m) &\leq G(b_n, b_{n+1}, b_{n+1}) + G(b_{n+1}, b_{n+2}, b_{n+2}) \\ &+ G(b_{n+2}, b_{n+3}, b_{n+3}) + ... + G(b_{m-1}, b_m, b_m) \\ &\leq (\beta^n + \beta^{n+1} + ... + \beta^{m-1}) G(b_0, b_1, b_1) \\ &\leq \frac{\beta^n}{1-\beta} G(b_0, b_1, b_1) \end{aligned}$$

taking limit as $n, m \to \infty$, we get $lim_{n,m\to\infty} G(b_n, b_m, b_m) = 0$. $\therefore$ this shows that $\{b_n\}$ is a G-Cauchy sequence in X.Since given (X, G) is G-Complete metric space. $\therefore$,there exists a point $x \in X$ s.t. $lim_{n\to\infty} b_n = x$ and $\therefore lim_{n\to\infty} b_n = lim_{n\to\infty} Sa_n = lim_{n\to\infty} Ta_{n+1} = x$. Since the mapping S or T is Continuous.Suppose T is continuous , $\therefore lim_{n\to\infty} TSa_n = Tx$. also given that S and T are compatible. $\therefore lim_{n\to\infty} G(TSa_n, STa_n, STa_n) = 0$ .This gives $lim_{n\to\infty} STa_n = Tx$ From (3) we get

$$G(STa_n, Sa_n, Sa_n) \leq \beta G(TTa_n, Ta_n, Ta_n)$$

taking limit as $n \to \infty$, we get $Tx = x$ Again from (3)we get

$$G(Sa_n, Sx, Sx) \leq \beta G(Ta_n, Tx, Tx)$$

taking limit as $n \to \infty$, we get $Tx = x$ $\therefore$ we get $Tx = Sx = x$ .Hence x is a common fixed point of S and T. For Uniqueness ,If possible suppose let x_1 be another common fixed point of S and T. Then we have, $G(x, x_1, x_1) > 0$ and

$$\begin{aligned} G(x, x_1, x_1) &= G(Sx, Sx_1, Sx_1) \\ &\leq \beta G(Tx, Tx_1, Tx_1) \\ &= \beta G(x, x_1, x_1) \\ &< G(x, x_1, x_1) \end{aligned}$$

which is impossible. $\therefore$ $x = x_1$ .Hence uniqueness follows. □

Example 2.16. *If $X = [-1, 1]$ and G be a G-metric space s.t. $G : X^3 \to R^+$ defined by*

$$G(x_1, y_1, z_1) = (|x_1 - y_1| + |y_1 - z_1| + |z_1 - x_1|)$$

for all $x_1, y_1, z_1 \in X$ .Then X is a G-Metric space.We define $S(x_1) = \frac{x_1}{6}$ and $T(x_1) = \frac{x_1}{2}$.
If S is Continuous and $S(X) \subseteq T(X)$.
Here $G(Sx_1, Sy_1, Sz_1) \leq \beta G(Tx_1, Ty_1, Tz_1)$ is true for all $x_1, y_1, z_1 \in X$, $\frac{1}{3} \leq \beta < 1$ and 0 is the common fixed point of S and T which is Unique.

In 2017,Latpate V.V. and Dolhare U.P [10] proved common fixed point theorem for pair of compatible maps in G-Metric space.

Theorem 2.17. *Let X be a complete G-metric space. $S, T : X \to X$ be two compatible maps on X and which satisfies the following conditions,*
(i) $S(X) \subseteq T(X)$,
(ii)S or T is G-continuous,
(iii) $G(Sa, Sb, Sc) \leq \alpha G(Sa, Tb, Tc) + \beta G(Ta, Sb, Tc) + \gamma G(Ta, Tb, Sc) + \delta G(Sa, Tb, Tc)$
for every a,b,c in X and $\alpha, \beta, \gamma, \delta \geq 0$ *with* $0 \leq \alpha + 3\beta + 3\gamma + 3\delta < 1$ *. Then S and T have unique common fixed point in X.*

Now,we prove our Main result,for the compatible maps.

3 Main Result

Theorem 3.1. *Let* (X, G) *be a complete G-Metric space,Let S and T be self mappings of X satisfying the following conditions,*

1. $S(X) \subseteq T(X)$
2. *S or T is continuous,*
3. $$G(Sa, Sb, Sc) \leq k \ max \begin{Bmatrix} G(Ta, Tb, Tc), G(Ta, Sa, Sa), G(Ta, Sb, Sb) \\ G(Ta, Sc, Sc), G(Tb, Sb, Sb), G(Tb, Sa, Sa) \\ G(Tb, Sc, Sc), G(Tc, Sc, Sc), G(Tc, Sa, Sa) \\ G(Tc, Sb, Sb) \end{Bmatrix}$$

for all $a, b, c \in X$ *, where* $0 \leq k < \frac{1}{4}$ *.Then S and T have unique common fixed point in X.Provided S and T are compatible maps.*

Proof. Let a_0 be an arbitrary point in X.By,using equation (1),one can choose a point a_1 in X s.t. $Sa_0 = Ta_1$ .In general we can choose a point a_{n+1} s.t. $b_n = Sa_n = Ta_{n+1}$, $n =$

$0, 1, 2, 3, ...$ from (3), we have
(3.1)

$$G(Sa_n, Sa_{n+1}, Sa_{n+1}) \leq k\ max \left\{ \begin{array}{c} G(Ta_n, Ta_{n+1}, Ta_{n+1}), G(Ta_n, Sa_n, Sa_n), G(Ta_n, Sa_{n+1}, Sa_{n+1}), \\ G(Ta_n, Sa_{n+1}, Sa_{n+1}), G(Ta_{n+1}, Sa_{n+1}, Sa_{n+1}), G(Ta_{n+1}, Sa_n, Sa_n), \\ G(Ta_{n+1}, Sa_{n+1}, Sa_{n+1}), G(Ta_{n+1}, Sa_{n+1}, Sa_{n+1}), G(Ta_{n+1}, Sa_n, Sa_n), \\ G(Ta_{n+1}, Sa_{n+1}, Sa_{n+1}) \end{array} \right\}$$

or

$$G(b_n, b_{n+1}, b_{n+1}) \leq k\ max\{G(b_{n-1}, b_n, b_n), G(b_{n-1}, b_{n+1}, b_{n+1}), G(b_n, b_{n+1}, b_{n+1})\}$$

Possibility 1 If

$$max\{G(b_{n-1}, b_n, b_n), G(b_{n-1}, b_{n+1}, b_{n+1}), G(b_n, b_{n+1}, b_{n+1})\} = G(b_{n-1}, b_n, b_n)$$

then using (3), we get

$$G(b_n, b_{n+1}, b_{n+1}) \leq kG(b_{n-1}, b_n, b_n)$$

, continuing in the same way, we have

$$G(b_n, b_{n+1}, b_{n+1}) \leq k^n G(b_0, b_1, b_1)$$

.Therefore for all $n, m \in N \quad n < m$ and by using rectangle inequality, we get

$$\begin{aligned} G(b_n, b_m, b_m) &\leq G(b_n, b_{n+1}, b_{n+1}) + G(b_{n+1}, b_{n+2}, b_{n+2}) \\ &+ ...... + G(b_{m-1}, b_m, b_m) \\ &\leq (k^n + k^{n+1} + ... + k^{m-1})G(b_0, b_1, b_1) \\ &\leq \frac{k^n}{1-k} G(b_0, b_1, b_1) \end{aligned}$$

taking limit as $n, m \to \infty$, we have $lim_{n\to\infty} G(b_n, b_m, b_m) = 0$ .Thus $\{b_n\}$ is a G-Cauchy sequence in X.

Possibility2 If

$$max\{G(b_{n-1}, b_n, b_n), G(b_{n-1}, b_{n+1}, b_{n+1}), G(b_n, b_{n+1}, b_{n+1})\} = G(b_{n-1}, b_{n+1}, b_{n+1})$$

from (3) and using rectangle inequality,we get

$$\begin{aligned} G(b_n, b_{n+1}, b_{n+1}) &\leq k\ G(b_{n-1}, b_{n+1}, b_{n+1}) \\ &\leq k(G(b_{n-1}, b_n, b_n) + G(b_n, b_{n+1}, b_{n+1})) \end{aligned}$$

this gives

$$G(b_n, b_{n+1}, b_{n+1}) \leq \frac{k}{1-k} G(b_{n-1}, b_n, b_n)$$

i.e.

$$G(b_n, b_{n+1}, b_{n+1}) \leq \beta G(b_{n-1}, b_n, b_n)$$

, where $\beta = \frac{k}{1-k}$ and $\beta < 1$ as $0 \leq k < \frac{1}{4}$ using possibility (1), we have $\{b_n\}$ is a G-Cauchy sequence in X.

Possibility3 If

$$max\ \{G(b_{n-1}, b_n, b_n), G(b_{n-1}, b_{n+1}, b_{n+1}), G(b_n, b_{n+1}, b_{n+1})\} = G(b_n, b_{n+1}, b_{n+1})$$

then from (3) , we have

$$G(b_n, b_{n+1}, b_{n+1}) \leq\ kG(b_n, b_{n+1}, b_{n+1})$$

which is a contradiction, since $k < \frac{1}{4}$. Therefore in all cases the sequence $\{b_n\}$ is a G-Cauchy sequence in X.Since (X, G) is G-complete Metric space.$\therefore$, thee exists a point $x \in X$ s.t. $lim_{n\to\infty} b_n = x$, we have

$$lim_{n\to\infty} b_n = lim_{n\to\infty} Sa_n = lim_{n\to\infty} Ta_{n+1} = x$$

. since one of the maps S or T is continuous.Suppose we can assume that T is continuous, therefore $lim_{n\to\infty} TSa_n = lim_{n\to\infty} TTa_n = Tx$ Also given that S and T are compatible.Therefore, $lim_{n\to\infty} G(TSa_n, STa_n, STa_n) = 0$ gives $lim_{n\to\infty} STa_n = Tx$,Now we claim that $Tx = x$, from (3) we have

(3.2)

$$G(STa_n, Sa_n, Sa_n) \leq k\, max \left\{ \begin{array}{c} G(TTa_n, Ta_n, Ta_n), G(TTa_n, STa_n, STa_n), G(TTa_n, Sa_n, Sa_n) \\ G(TTa_n, Sa_n, Sa_n), G(Ta_n, Sa_n, Sa_n), G(Ta_n, STa_n, STa_n) \\ G(Ta_n, Sa_n, Sa_n), G(Ta_n, Sa_n, Sa_n), G(Ta_n, STa_n, STa_n) \\ G(Ta_n, Sa_n, Sa_n) \end{array} \right\}$$

taking, limit as $n \to \infty$ and by using proposition (2.11), we have

$$\begin{aligned} G(Tx,x,x) &\leq k\ max\{G(Tx,x,x), G(x,Tx,Tx)\} \\ &\leq k\ max\{G(Tx,x,x), 2G(Tx,x,x)\} \\ &= 2kG(Tx,x,x) \end{aligned}$$

which is a contradiction since $k < \frac{1}{4}$. Hence $Tx = x$, similarly we will show that $Tx = Sx = x$,for that we put $a = a_n, b = c = x$ in (3), we get

$$(3.3)\quad G(Sa_n,Sx,Sx) \leq k\ max \left\{ \begin{array}{c} G(Ta_n,Tx,Tx), G(Ta_n,Sa_n,Sa_n), G(Ta_n,Sx,Sx) \\ G(Ta_n,Sx,Sx), G(Tx,Sx,Sx), G(Tx,Sa_n,Sa_n) \\ G(Tx,Sx,Sx), G(Tx,Sx,Sx), G(Tx,Sa_n,Sa_n) \\ G(Tx,Sx,Sx) \end{array} \right\}$$

taking limit as $n \to \infty$, we get

$$G(x,Sx,Sx) \leq kG(x,Sx,Sx)$$

, which is a contradiction since $k < \frac{1}{4}$.Hence $Sx = Tx = x$.Thus x is a common fixed point of S and T.

To prove Uniqueness, we assume that $x_1 \neq x$ be another common fixed point of S and T.Then $G(x_1,x,x) > 0$

(3.4)

$$G(x_1,x,x) = G(Sx_1,Sx,Sx) \leq k\ max \left\{ \begin{array}{c} G(Tx_1,Tx,Tx), G(Tx_1,Sx_1,Sx_1), G(Tx_1,Sx,Sx) \\ G(Tx_1,Sx,Sx), G(Tx,Sx,Sx), G(Tx,Sx_1,Sx_1) \\ G(Tx,Sx,Sx), G(Tx,Sx,Sx), G(Tx,Sx_1,Sx_1) \\ G(Tx,Sx,Sx) \end{array} \right\}$$

By proposition (2.11), we get

$$\begin{aligned} G(x_1,x,x) &\leq k\ max\{G(x_1,x,x), G(x,x_1,x_1)\} \\ &\leq k\ max\{G(x_1,x,x), 2G(x_1,x,x)\} \\ &= 2kG(x_1,x,x), \end{aligned}$$

which is a contradiction since $k < \frac{1}{4}$ □

Example 3.2. *Let* $X = [-1, 1]$ *and let* (X, G) *be a G-metric on X.G-metric function is defined as* $G(a_1, b_1, c_1) = (|a_1 - b_1| + |b_1 - c_1| + |c_1 - a_1|)$ *,for all* $a_1, b_1, c_1 \in X$*. Then* (X, G) *be a G-metric space and define* $Sx = \frac{x}{9}$ *and* $Tx = x$ *,then* $S(X) \subseteq T(X)$ *.Also inequality (3) satisfies for all* $a_1, b_1, c_1 \in X$*. and 0 is the unique common fixed point of S and T.*

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