Some Expansions for a Multivariable Gimel-Function

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ABSTRACT
The aim of the present paper is to evaluate two finite integrals involving the product of trigonometric function and the multivariable Gimel-function. These integrals have been utilized to derive the expansion formula for generalized multivariable Gimel-function in series involving trigonometric function.

KEYWORDS : Multivariable Gimel-function, multiple integral contours, expansion serie, finite integrals.

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1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C},\mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0=\mathbb{N}\cup\{0\}$.

The subject of expansion formulae and Fourier series of special functions occupies a large place in the literature of special functions. Certain expansion formulae and Fourier series of trigonometric functions play an important role in the development of the theories of special functions . In this paper, we establish two single expansion formula for multivariable Gimel-function.

We define a generalized transcendental function of several complex variables noted $\gimel$.

$$\gimel(z_1,\cdots,z_r)=\gimel^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|}\mathrm{z}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\end{array}\right.$$

$$\begin{array}{c}[(\mathrm{a}_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}};[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},\\ [\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{1,q_{i_2}};\end{array}$$

$$\begin{array}{ll}[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots; & [(\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],\\ [\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{1,q_{i_3}};\cdots; & \quad\cdots\end{array}$$

$$\begin{array}{ll}[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{n_r+1,p_r}]: & [(\mathrm{c}^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]\\ [\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{1,q_r}]: & [(\mathrm{d}^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]\end{array}$$

$$\left.\begin{array}{l};\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})_{m^{(r)}+1,p^{(r)}_i}]\\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})_{n^{(r)}+1,q^{(r)}_i}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha_{2ji_2}^{(k)}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta_{2ji2}^{(k)}s_k)]}$$

$$\frac{\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)}]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $n_2,\cdots,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2,\cdots,0\leqslant m_r,0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$

$0\leqslant n^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant p_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)} \in \mathbb{C}; (j=1,\cdots,n^{(k)}); (k=1,\cdots,r); d_j^{(k)} \in \mathbb{C}; (j=1,\cdots,m^{(k)}); (k=1,\cdots,r)$.

$a_{kji_k} \in \mathbb{C}; (j=n_k+1,\cdots,p_{i_k}); (k=2,\cdots,r)$.

$b_{kji_k} \in \mathbb{C}; (j=1,\cdots,q_{i_k}); (k=2,\cdots,r)$.

$d_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i=1,\cdots,R^{(k)}); (j=m^{(k)}+1,\cdots,q_{i^{(k)}}); (k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i=1,\cdots,R^{(k)}); (j=n^{(k)}+1,\cdots,p_{i^{(k)}}); (k=1,\cdots,r)$.

The contour L_k is in the $s_k(k=1,\cdots,r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k\right)(j=1,\cdots,n_2), \Gamma^{A_{3j}}\left(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k\right)$ $(j=1,\cdots,n_3),\cdots,\Gamma^{A_{rj}}\left(1-a_{rj}+\sum_{i=1}^{r}\alpha_{rj}^{(i)}\right)(j=1,\cdots,n_r)$, $\Gamma^{C_j^{(k)}}\left(1-c_j^{(k)}+\gamma_j^{(k)}s_k\right)(j=1,\cdots,n^{(k)})(k=1,\cdots,r)$to the right of the contour L_k and the poles of $\Gamma^{D_j^{(k)}}\left(d_j^{(k)}-\delta_j^{(k)}s_k\right)(j=1,\cdots,m^{(k)})(k=1,\cdots,r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2}A_i^{(k)}\pi$ where

$$A_i^{(k)} = \sum_{j=1}^{m^{(k)}} D_j^{(k)}\delta_j^{(k)} + \sum_{j=1}^{n^{(k)}} C_j^{(k)}\gamma_j^{(k)} - \tau_{i^{(k)}}\left(\sum_{j=m^{(k)}+1}^{q_i^{(k)}} D_{ji^{(k)}}^{(k)}\delta_{ji^{(k)}}^{(k)} + \sum_{j=n^{(k)}+1}^{p_i^{(k)}} C_{ji^{(k)}}^{(k)}\gamma_{ji^{(k)}}^{(k)}\right) +$$

$$-\tau_{i_2}\left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2}\alpha_{2ji_2}^{(k)} + \sum_{j=1}^{q_{i_2}} B_{2ji_2}\beta_{2ji_2}^{(k)}\right) - \cdots - \tau_{i_r}\left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r}\alpha_{rji_r}^{(k)} + \sum_{j=1}^{q_{i_r}} B_{rji_r}\beta_{rji_r}^{(k)}\right) \tag{1.4}$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,), max(\,|z_1|,\cdots,|z_r|\,) \rightarrow 0$

$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,), min(\,|z_1|,\cdots,|z_r|\,) \rightarrow \infty$ where $i=1,\cdots,r$:

$\alpha_i = \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]$ and $\beta_i = \max_{1\leqslant j\leqslant n^{(i)}} Re\left[C_j^{(i)}\left(\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)\right]$

Remark 1.
If $n_2=\cdots=n_{r-1}=p_{i_2}=q_{i_2}=\cdots=p_{i_{r-1}}=q_{i_{r-1}}=0$ and $A_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=A_{rji_r}=B_{rji_r}=1$ $A_{rj}=A_{rji_r}=B_{rji_r}=1$, then the multivariable Gimel-function reduces in the multivariable Aleph- function defined by Ayant [1].

Remark 2.
If $n_2=\cdots=n_r=p_{i_2}=q_{i_2}=\cdots=p_{i_r}=q_{i_r}=0$ and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2=\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [7].

Remark 3.
If$A_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=A_{rji_r}=B_{rji_r}=1$and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2=\cdots=R_r=R^{(1)}$ $=\cdots=R^{(r)}=1$, then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [6].

Remark 4.

If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H-function defined by Srivastava and panda [8,9].

In your investigation, we shall use the following notations.

$$\mathbb{A}=[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.5}$$

$$\mathbf{A}=[(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A=[(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \tag{1.7}$$

$$\mathbb{B}=[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})_{1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B}=[\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{1,q_{i_r}}] \tag{1.9}$$

$$\mathrm{B}=[(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}];\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \tag{1.10}$$

$$U=0,n_2;0,n_3;\cdots;0,n_{r-1};V=m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X=p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

2. Required results.

We require the following relations for the development of the present work.

Lemma 1. (Luke [5], p. 3, 1.2(15))

$$\int_0^\pi e^{\omega v\theta}\sin^\alpha\theta\mathrm{d}\theta=\frac{\pi e^{\frac{1}{2}\omega v\pi}\Gamma(\alpha+1)}{2^\alpha\Gamma\left(\frac{\alpha\pm v}{2}+1\right)} \tag{2.1}$$

provided $Re(\alpha)>-1$.

Orthogonality of exponential function ([4], p. 62)

Lemma 2.

$$\int_a^b exp\left(\frac{2m\omega\pi x}{a-b}\right)exp\left(\frac{2n\omega\pi x}{a-b}\right)\mathrm{d}x=\left[\begin{array}{c}0 \text{ if m}\neq n\\ .\\ \text{b-a if m}=\text{n}\end{array}\right. \tag{2.2}$$

3. Main integral.

In this section, we evaluate two finite integrals

Theorem 1.

$$\int_0^{\pi} \sin^{2\alpha}\theta \gimel\left(z_1 e^{\omega\delta_1\theta}\sin^{2\mu_1}\theta,\cdots,z_r e^{\omega\delta_r\theta}\sin^{2\mu_r}\theta\right)\mathrm{d}\theta = \frac{\pi e^{\omega\pi\upsilon}}{4^{\alpha}}$$

$$\gimel^{U;0,n_r+1:V}_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|r} z_1\frac{e^{\omega\delta_1\pi}}{4^{\mu_1}} & \mathbb{A};(-2\alpha;2\mu_1,\cdots,2\mu_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\frac{e^{\omega\delta_r\pi}}{4^{\mu_r}} & \mathbb{B};\mathbf{B},(\upsilon-\alpha;\mu_1-\delta_1,\cdots,\mu_r-\delta_r;1),(-\upsilon-\alpha;\mu_1+\delta_1,\cdots,\mu_r+\delta_r;1):B \end{array}\right) \tag{3.1}$$

where $0\leqslant\ \delta_i<\mu_i(i=1,\cdots,r)$

$$\int_0^{\pi} \sin^{2\alpha}\theta \gimel\left(z_1 e^{\omega\delta_1\theta}\sin^{2\mu_1}\theta,\cdots,z_r e^{\omega\delta_r\theta}\sin^{2\mu_r}\theta\right)\mathrm{d}\theta = \frac{\pi e^{\omega\pi\upsilon}}{4^{\alpha}}$$

$$\gimel^{U;0,n_r+1:V}_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|r} z_1\frac{e^{\omega\delta_1\pi}}{4^{\mu_1}} & \mathbb{A};(-2\alpha;2\mu_1,\cdots,2\mu_r;1),\mathbf{A},(1+\alpha-\upsilon;\delta_1-\mu_1,\cdots,\delta_r-\mu_r;1):A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\frac{e^{\omega\delta_r\pi}}{4^{\mu_r}} & \mathbb{B};\mathbf{B},(-\upsilon-\alpha;\delta_1+\mu_1,\cdots,\delta_r+\mu_r;1):B \end{array}\right) \tag{3.2}$$

where $0\leqslant\mu_i<\delta_i(i=1,\cdots,r)$

$$\int_0^{\pi} \sin^{2\alpha}\theta \gimel\left(z_1 e^{\omega\delta_1\theta}\sin^{2\delta_1}\theta,\cdots,z_r e^{\omega\delta_r\theta}\sin^{2\delta_r}\theta\right)\mathrm{d}\theta = \frac{\pi e^{\omega\pi\upsilon}}{4^{\alpha}\Gamma(1+\alpha-\upsilon)}$$

$$\gimel^{U;0,n_r+1:V}_{X;p_{i_r}+1,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|r} z_1\frac{e^{\omega\delta_1\pi}}{4^{\delta_1}} & \mathbb{A};(-2\alpha;2\delta_1,\cdots,2\delta_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\frac{e^{\omega\delta_r\pi}}{4^{\delta_r}} & \mathbb{B};\mathbf{B},(-\upsilon-\alpha;2\delta_1,\cdots,2\delta_r;1):B \end{array}\right) \tag{3.3}$$

where $\mu_i=\delta_i>0(i=1,\cdots,r)$, provided

$Re(1+2\alpha)+\sum_{i=1}^{r}\mu_i\min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0, Re(\alpha)>-\frac{1}{2}$ and

$\left|arg\left(z_i e^{\omega\delta_i\theta}\sin^{2\mu_i}\theta\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Proof

To establish the theorem 1, expressing the multivariable Gimel-function in the Mellin-Barnes multiple integrals contour with the help of (1.1) and interchanging the order of integrations, which is justified under the conditions mentioned above, we get

$$\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\left[\int_0^{\pi}e^{2\omega(\upsilon+\sum_{i=1}^{r}\delta_i s_i-1)}(\sin\theta)^{2(\alpha+\sum_{i=1}^{r}\mu_i s_i)}\mathrm{d}\theta\right]\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.4}$$

Evaluating the inner integral with the help of lemma 1 and Interpreting the resulting expression with the help of (1.1), we obtain the desired theorem 1.

Theorem 2.

$$\int_0^{\pi} 4^{\omega\upsilon\theta}\sin^{2\alpha}\theta\beth\left(z_1 e^{-2\omega\delta_1\theta}\sin^{2\mu_1}\theta,\cdots,z_r e^{-2\omega\delta_r\theta}\sin^{2\mu_r}\theta\right)\mathrm{d}\theta=\frac{\pi e^{\omega\pi\upsilon}}{4^{\alpha}}$$

$$\beth^{U;0,n_r+1:V}_{X;p_{i_r}+1,q_{i_r}+2,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1\frac{e^{-\omega\delta_1\pi}}{4^{\mu_1}} & \mathbb{A};(-2\alpha;2\mu_1,\cdots,2\mu_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\frac{e^{-\omega\delta_r\pi}}{4^{\mu_r}} & \mathbb{B};\mathbf{B},(\upsilon-\alpha;\mu_1+\delta_1,\cdots,\mu_r+\delta_r;1),(-\upsilon-\alpha;\mu_1-\delta_1,\cdots,\mu_r-\delta_r;1):B \end{array}\right) \tag{3.5}$$

where $0\leqslant \delta_i<\mu_i(i=1,\cdots,r)$

$$\int_0^{\pi} 4^{\omega\upsilon\theta}\sin^{2\alpha}\theta\beth\left(z_1 e^{-2\omega\delta_1\theta}\sin^{2\mu_1}\theta,\cdots,z_r e^{-2\omega\delta_r\theta}\sin^{2\mu_r}\theta\right)\mathrm{d}\theta=\frac{\pi e^{\omega\pi\upsilon}}{4^{\alpha}}$$

$$\beth^{U;0,n_r+1:V}_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1\frac{e^{-\omega\delta_1\pi}}{4^{\mu_1}} & \mathbb{A};(-2\alpha;2\mu_1,\cdots,2\mu_r;1),\mathbf{A},(1+\alpha+\upsilon;\delta_1-\mu_1,\cdots,\delta_r-\mu_r;1):A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\frac{e^{-\omega\delta_r\pi}}{4^{\mu_r}} & \mathbb{B};\mathbf{B},(\upsilon-\alpha;\delta_1+\mu_1,\cdots,\delta_r+\mu_r;1):B \end{array}\right) \tag{3.6}$$

where $0\leqslant \mu_i<\delta_i(i=1,\cdots,r)$

$$\int_0^{\pi} 4^{\omega\upsilon\theta}\sin^{2\alpha}\theta\beth\left(z_1 e^{-2\omega\delta_1\theta}\sin^{2\mu_1}\theta,\cdots,z_r e^{-2\omega\delta_r\theta}\sin^{2\mu_r}\theta\right)\mathrm{d}\theta=\frac{\pi e^{\omega\pi\upsilon}}{4^{\alpha}\Gamma(1+\alpha+\upsilon)}$$

$$\beth^{U;0,n_r+1:V}_{X;p_{i_r}+1,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1\frac{e^{-\omega\delta_1\pi}}{4^{\delta_1}} & \mathbb{A};(-2\alpha;2\delta_1,\cdots,2\delta_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\frac{e^{-\omega\delta_r\pi}}{4^{\delta_r}} & \mathbb{B};\mathbf{B},(\upsilon-\alpha;2\delta_1,\cdots,2\delta_r;1):B \end{array}\right) \tag{3.7}$$

where $\mu_i=\delta_i>0(i=1,\cdots,r)$, provided

$$Re(1+2\alpha)+\sum_{i=1}^{r}\mu_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0, Re(\alpha)>-\frac{1}{2}\text{ and}$$

$\left|arg\left(z_i e^{-\omega\delta_i\theta}\sin^{2\mu_i}\theta\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Similarly the theorem 2 can be established.

4. Expansion formulae.

In this section, we established expansions multivariable Gimel-function in series involving exponential functions and multivariable Gimel-function by using integrals evaluated in the above section and orthogonality property of exponential functions.

Theorem 3.

$$\sin^{2\alpha}\theta\beth\left(z_1 e^{2\omega\delta_1\theta}\sin^{2\mu_1}\theta,\cdots,z_r e^{2\omega\delta_r\theta}\sin^{2\mu_r}\theta\right)=4^{-\alpha}\sum_{u=-\infty}^{\infty}e^{\omega u(\pi-2\theta)}$$

$$\beth^{U;0,n_r+1:V}_{X;p_{i_r}+1,q_{i_r}+2,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1\frac{e^{\omega\delta_1\pi}}{4^{\mu_1}} & \mathbb{A};(-2\alpha;2\mu_1,\cdots,2\mu_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\frac{e^{\omega\delta_r\pi}}{4^{\mu_r}} & \mathbb{B};\mathbf{B},(u-\alpha;\mu_1-\delta_1,\cdots,\mu_r-\delta_r;1),(-u-\alpha;\mu_1+\delta_1,\cdots,\mu_r+\delta_r;1):B \end{array}\right) \tag{4.1}$$

where $0 \leqslant \delta_i < \mu_i (i = 1, \cdots, r)$

$$\sin^{2\alpha}\theta \gimel\left(z_1 e^{2\omega\delta_1\theta}\sin^{2\mu_1}\theta, \cdots, z_r e^{2\omega\delta_r\theta}\sin^{2\mu_r}\theta\right) = 4^{-\alpha}\sum_{u=-\infty}^{\infty} e^{\omega u(\pi-2\theta)}$$

$$\gimel^{U;0,n_r+1:V}_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|r} z_1\frac{e^{\omega\delta_1\pi}}{4^{\mu_1}} & \mathbb{A};\ (-2\alpha; 2\mu_1, \cdots, 2\mu_r; 1), \mathbf{A}, (1+\alpha-u; \delta_1-\mu_1, \cdots, \delta_r-\mu_r; 1) : A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\frac{e^{\omega\delta_r\pi}}{4^{\mu_r}} & \mathbb{B};\ \mathbf{B},(-u-\alpha; \delta_1+\mu_1, \cdots, \delta_r+\mu_r; 1) : B \end{array}\right) \tag{4.2}$$

where $0 \leqslant \mu_i < \delta_i (i = 1, \cdots, r)$

$$\sin^{2\alpha}\theta \gimel\left(z_1 e^{2\omega\delta_1\theta}\sin^{2\mu_1}\theta, \cdots, z_r e^{2\omega\delta_r\theta}\sin^{2\mu_r}\theta\right) = 4^{-\alpha}\sum_{u=-\infty}^{\infty} \frac{e^{\omega u(\pi-2\theta)}}{\Gamma(1+\alpha-u)}$$

$$\gimel^{U;0,n_r+1:V}_{X;p_{i_r}+1,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1\frac{e^{\omega\delta_1\pi}}{4^{\delta_1}} & \mathbb{A};\ (-2\alpha; 2\delta_1, \cdots, 2\delta_r; 1), \mathbf{A} : A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\frac{e^{\omega\delta_r\pi}}{4^{\delta_r}} & \mathbb{B};\ \mathbf{B},(-u-\alpha; 2\delta_1, \cdots, 2\delta_r; 1) : B \end{array}\right) \tag{4.3}$$

where $\mu_i = \delta_i > 0 (i = 1, \cdots, r)$, under the same existence conditions that theorem 1.

Proof.
To prove the throrem 3, let

$$\sin^{2\alpha}\theta \gimel\left(z_1 e^{2\omega\delta_1\theta}\sin^{2\mu_1}\theta, \cdots, z_r e^{2\omega\delta_r\theta}\sin^{2\mu_r}\theta\right) = \sum_{u=-\infty}^{\infty} A_u e^{-2\omega u\theta} \tag{4.4}$$

The above equation is valid since the expression on the left-hand side is continuous and bounded variation in the interval $(0, \pi)$. Multiplying both sides of the equation (4.4) by $e^{2\omega v\theta}$ and integrating with respect to θ from 0 to π on the left-hand side using the theorem 1 and on the right-hand side changing the order of summation and integration in view ([3], p. 176(75)) and then applying orthogonality property of exponential function, see the lemma 2, we obtain

$$C_v = 4^{-\alpha}e^{\omega\pi v}$$

$$\gimel^{U;0,n_r+1:V}_{X;p_{i_r}+1,q_{i_r}+2,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|r} z_1\frac{e^{\omega\delta_1\pi}}{4^{\mu_1}} & \mathbb{A};\ (-2\alpha; 2\mu_1, \cdots, 2\mu_r; 1), \mathbf{A} : A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\frac{e^{\omega\delta_r\pi}}{4^{\mu_r}} & \mathbb{B};\ \mathbf{B},(v-\alpha; \mu_1-\delta_1, \cdots, \mu_r-\delta_r; 1), (-v-\alpha; \mu_1+\delta_1, \cdots, \mu_r+\delta_r; 1) : B \end{array}\right) \tag{4.5}$$

where $0 \leqslant \delta_i < \mu_i (i = 1, \cdots, r)$

$$C_v = 4^{-\alpha}e^{\omega\pi v}$$

$$\gimel^{U;0,n_r+1:V}_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|r} z_1\frac{e^{\omega\delta_1\pi}}{4^{\mu_1}} & \mathbb{A};\ (-2\alpha; 2\mu_1, \cdots, 2\mu_r; 1), \mathbf{A}, (1+\alpha-v; \delta_1-\mu_1, \cdots, \delta_r-\mu_r; 1) : A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\frac{e^{\omega\delta_r\pi}}{4^{\mu_r}} & \mathbb{B};\ \mathbf{B},(-v-\alpha; \delta_1+\mu_1, \cdots, \delta_r+\mu_r; 1) : B \end{array}\right) \tag{4.6}$$

where $0 \leqslant \mu_i < \delta_i (i = 1, \cdots, r)$

$C_v = 4^{-\alpha} e^{\omega\pi v}\Gamma(1+\alpha-v)$

$$\gimel^{U;0,n_r+1:V}_{X;p_{i_r}+1,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1\frac{e^{\omega\delta_1\pi}}{4^{\delta_1}} & \mathbb{A};\ (-2\alpha;2\delta_1,\cdots,2\delta_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\frac{e^{\omega\delta_r\pi}}{4^{\delta_r}} & \mathbb{B};\ \mathbf{B},(-v+\alpha;2\delta_1,\cdots,2\delta_r;1):B \end{array}\right) \tag{4.7}$$

where $\mu_i = \delta_i > 0 (i = 1, \cdots, r)$, under the same existence conditions that theorem 1.

Now, substituting (4.4) (respectively (4.5) and (4.6)) in (4.4) we obtain the theorem 3.

Theorem 4.

$$\sin^{2\alpha}\theta\gimel\left(z_1 e^{-2\omega\delta_1\theta}\sin^{2\mu_1}\theta,\cdots,z_r e^{-2\omega\delta_r\theta}\sin^{2\mu_r}\theta\right) = 4^{-\alpha}\sum_{u=-\infty}^{\infty} e^{\omega u(\pi-2\theta)}$$

$$\gimel^{U;0,n_r+1:V}_{X;p_{i_r}+1,q_{i_r}+2,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1\frac{e^{\omega\delta_1\pi}}{4^{\mu_1}} & \mathbb{A};\ (-2\alpha;2\mu_1,\cdots,2\mu_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\frac{e^{\omega\delta_r\pi}}{4^{\mu_r}} & \mathbb{B};\ \mathbf{B},(u-\alpha;\mu_1+\delta_1,\cdots,\mu_r+\delta_r;1),(-u-\alpha;\mu_1-\delta_1,\cdots,\mu_r-\delta_r;1):B \end{array}\right) \tag{4.8}$$

where $0 \leqslant \delta_i < \mu_i (i = 1, \cdots, r)$

$$\sin^{2\alpha}\theta\gimel\left(z_1 e^{-2\omega\delta_1\theta}\sin^{2\mu_1}\theta,\cdots,z_r e^{-2\omega\delta_r\theta}\sin^{2\mu_r}\theta\right) = 4^{-\alpha}\sum_{u=-\infty}^{\infty} e^{\omega u(\pi-2\theta)}$$

$$\gimel^{U;0,n_r+1:V}_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1\frac{e^{\omega\delta_1\pi}}{4^{\mu_1}} & \mathbb{A};\ (-2\alpha;2\mu_1,\cdots,2\mu_r;1),\mathbf{A},(1+\alpha+u;\delta_1-\mu_1,\cdots,\delta_r-\mu_r;1):A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\frac{e^{\omega\delta_r\pi}}{4^{\mu_r}} & \mathbb{B};\ \mathbf{B},(u-\alpha;\delta_1+\mu_1,\cdots,\delta_r+\mu_r;1):B \end{array}\right) \tag{4.9}$$

where $0 \leqslant \mu_i < \delta_i (i = 1, \cdots, r)$

$$\sin^{2\alpha}\theta\gimel\left(z_1 e^{-2\omega\delta_1\theta}\sin^{2\mu_1}\theta,\cdots,z_r e^{-2\omega\delta_r\theta}\sin^{2\mu_r}\theta\right) = 4^{-\alpha}\sum_{u=-\infty}^{\infty} \frac{e^{\omega u(\pi-2\theta)}}{\Gamma(1+\alpha+u)}$$

$$\gimel^{U;0,n_r+1:V}_{X;p_{i_r}+1,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1\frac{e^{\omega\delta_1\pi}}{4^{\delta_1}} & \mathbb{A};\ (-2\alpha;2\delta_1,\cdots,2\delta_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\frac{e^{\omega\delta_r\pi}}{4^{\delta_r}} & \mathbb{B};\ \mathbf{B},(u-\alpha;2\delta_1,\cdots,2\delta_r;1):B \end{array}\right) \tag{4.10}$$

where $\mu_i = \delta_i > 0 (i = 1, \cdots, r)$, under the same existence conditions that theorem 2.

Similarly the theorem 4 can be established by starting with the relation :

$$\sin^{2\alpha}\theta\gimel\left(z_1 e^{-2\omega\delta_1\theta}\sin^{2\mu_1}\theta,\cdots,z_r e^{-2\omega\delta_r\theta}\sin^{2\mu_r}\theta\right) = \sum_{u=-\infty}^{\infty} A_u e^{-2\omega u\theta} \tag{4.11}$$

and utilising the theorem 2 instead of theorem 1.

5. Conclusion.

The importance of our all the results lies in their manifold generality. By specialising the various parameters as well as variables in the multivariable Gimel-function, we get a several expansion formulae involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several intersting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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