Study of Common Fixed Point Results for Compatible Mappings in Cone Metric Space

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Abstract: - *In this paper, we establish the common fixed point theorems of weakly compatible mappings in cone metric space. Our results extend, modify and improve various well known results in the literature.*

Keywords: -*Fixed point, common fixed point, cone metric space, compatible mapping.*

I. INTRODUCTION

The concept of fixed point theory was firstly propounded by Poincare in 1886. After that in 1912, Brower proved fixed point theorem for solution of equation f(x) = x.

In 1922, Polish mathematician Stefan Banach [2] who was one of the founders of functional analysis gave a principle which was one of the fundamental principles in the field of functional analysis.

In 1976, Jungck [5] proved a common fixed point theorem for commutative mappings, generalizing the famous Banach contraction principle. Sesa [13] introduced the notion of weakly commutative maps. Also, in 1986,Jungck[6] introduced the notion of compatible mapping in order to generalize the concept of weak commutative. Again, Pant [9, 10] defined R-weakly commutating maps and established some common fixed point theorems, assuming the continuity of at least one of the mappings. In 1997, H. K. Pathak and M. S. Khan [12] compared various types of compatible maps and gave a new common fixed points result. Kannan[8, 9] proved the existence of fixed point for a mapping that can have a discontinuity in a domain; however maps involved in each case were continuous at the fixed point.

In 1998, Jungck and Rhoades [7] defined a pair of self mappings to be weakly compatible if the commutes at their coincidence points. Then, applying these concepts, several authors have obtained coincidence point results for various classes of mappings in a metric space. On the other hand, Huang and Zhang [3] generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions. In 2008, Abbas and Jungck[1] proved some common fixed point theorems for weakly compatible mappings in the setting of cone metric space; also K. Jha [4] proved common fixed point theorems in a cone metric space. In 2013,S. k. Tiwari, R. P. Dubey[14] proved some fixed point theorems for generalized contractive mapping in cone metric space and with A. K. Dubey [15] they gave common fixed point results in cone metric spaces.Recently, S. K. Tiwari, and Kaushik Das[16] extended some common fixed point results for contractive mappings in cone metric spaces.

The aim of this paper is to establishcommon fixed point theorems for a pair of weakly compatible mappings in a cone metric space.

II. PRELIMINARIES

We recall some definitions and properties of cone metric spaces[3].

***Definition 2.1*[3]:-** Let E be a real Banach space and P be a subset of E. Then P is called **cone**iff,
(i) P is closed and nonempty subset of E and $P \neq \{0\}$;
(ii) $a, b \in R^{+}, x, y \in P \Rightarrow ax + by \in P$;

(iii) $x \in P$

Given a cone $P \subseteq E$. We define a partial ordering $\leq$ on E with respect to P by $x \leq y \Leftrightarrow y - x \in P$. We shall write $x << y$ if $y - x \in \operatorname{int} P$. If $\operatorname{int} P \neq \phi$, then cone P will be solid. The cone P is said to be **normal** if there is a number K>0 such that for all $x, y \in E$

$0 \leq x \leq y \Rightarrow \|x\| \leq k \|y\|$.

The least positive number K satisfying the above is called the normal constant of P.

***Definition 2.2*[3] : -** Let X is a non empty set. Let $d : X \times X \to E$ be a mapping satisfies

d_1 : $0 < d(x, y)$ And $d(x, y) = 0 \Leftrightarrow x = y$ for all $x, y \in X$

d_2 : $d(x, y) = d(y, x)$ For all $x, y \in X$

d_3 : $d(x, y) \leq d(x, z) + d(z, x)$ For all $x, y, z \in X$

Then d is called cone metric on X and (X, d) is called **cone metric space.**

Example: Let E=R^2, $P = \{(x, y) \in E : x, y \geq 0\} \subset R^2$ X=R and $d : X \times X \to R$ such that $d(x, y) = (|x - y|, \alpha |x - y|)$, where α≥0 is a constant. Then (X, d) is cone metric space

***Definition 2.3*: -** Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}_{n \geq 1}$ be a sequence in X. Then,

(1) $\{x_n\}_{n \geq 1}$ converges to x whenever for every $c \in E$ with $\theta << c$. If there is a natural number N such that $d(x_n, x) << c$ for all $n \geq N$. We devote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(2) $\{x_n\}_{n \geq 1}$ is said to be **Cauchy sequence** if for every $c \in E$ with $\theta << c$. if there is a natural number N such that $d(x_n, x_m) << c$ for all $n, m \geq N$.

(3) (X, d) is called a **complete cone metric space** if every Cauchy sequence in X is Convergent.

***Definition 2.4* : --** Let f and g are two self maps on a set . If fx = gx for some x in X then x is called **coincidence point** of f and g.

***Definition 2.5*: -** Two self mappings f and g of a metric space (X, d) are called **compatible** if

$$\lim_{n \to \infty} d(fgx_n, gfx_n) = 0$$

Whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t$ for some t in X.

***Definition 2.6*:** Let S and T be mappings from a metric space (X, d) into itself. S and T are said to be **weaklycompatible** if they commute at their coincidence point.

i.e. $Sx = Tx$ for some $x \in X \Rightarrow STx = TSx$.

***Preposition 2.7*:** Let f and g be weakly compatible self mappings of a set X. if f and g have a unique point of coincidence, that is $w = fx = gx$, then w is the unique common fixed point of f and g.

III . MAIN RESULTS:

Theorem 3.1: Let (X, d) be a cone metric space and P be a normal cone with normal constant k. Suppose the mappings f, g and h on X satisfying the contractive condition

$d(fx, gy) \leq \lambda[d(fx, hx) + d(gy, hy) + d(fx, hy) + d(hx, gy) + d(hx, hy)]$

If $f(x) \cup g(x) \subseteq h(x)$ and h(x) is a complete subspace of X. Then the maps f, g and h have a unique point of coincidence in X. Moreover, if (f, h) and (g, h) are weakly compatible pairs; then f, g and h have common unique fixed point in X.

Proof: suppose since $f(x) \cup g(x) \subseteq h(x)$, starting with x_0 we define a sequence $\{y_k\}$ such that

$y_{2k} = fx_{2k} = hx_{2k+1}$

and $y_{2k+1} = gx_{2k+1} = hx_{2k+2}$

Consider that,

$$d(y_{2k}, y_{2k+1}) = d(fx_{2k}, gx_{2k+1})$$

$$\Rightarrow d(y_{2k}, y_{2k+1}) \leq \frac{3\lambda}{1-2\lambda} d(y_{2k-1}, y_{2k})$$

$$\leq \lambda[d(fx_{2k}, hx_{2k}) + d(gx_{2k+1}, hx_{2k+1}) + d(fx_{2k}, hx_{2k+1}) + d(hx_{2k}, gx_{2k+1}) + d(hx_{2k}, hx_{2k+1})]$$

$$= \lambda[d(y_{2k}, y_{2k-1}) + d(y_{2k+1}, y_{2k}) + d(y_{2k+1}, y_{2k}) + d(y_{2k-1}, y_{2k+1}) + d(y_{2k-1}, y_{2k})]$$

$$\therefore d(y_{2k}, y_{2k+1}) \leq h d(y_{2k-1}, y_{2k})$$

Where $h = \frac{3\lambda}{1-2\lambda} < 1.$

Similarly we can show that

$d(y_{2k+2}, y_{2k+1}) \leq h' d(y_{2k+1}, y_{2k})$ Where $h' = \frac{3\lambda}{1-\lambda} < 1.$

Therefore, for all K,

$$d(y_{2k+2}, y_{2k+1}) \leq h d(y_{2k+1}, y_{2k})$$

$$\leq h^2 d(y_{2k}, y_{2k-1})$$

$\leq$ …………………..

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$$\leq h^{2k+1} d(y_0, y_1).$$

Now for any m>k,

$$d(y_{2k}, y_{2m}) \leq d(y_k, y_{2k+1}) + d(y_{2k+1}, y_{2k+2}) + \dots\dots\dots\dots + d(y_{2m-1}, y_{2m})$$

$$\leq h^{2k} d(y_0, y_1) + h^{2k+1} d(y_0, y_1) + \dots\dots\dots\dots + h^{2m-1} d(y_0, y_1)$$

$$\leq (h^{2k} + h^{2k+1} + h^{2k+2} \dots\dots\dots\dots) d(y_0, y_1)$$

$$= \frac{h^{2k}}{1-h} d(y_0, y_1).$$

Since P is normal with normal constant k.

Therefore, $\|d(y_{2k}, y_{2m}\| \leq \frac{h^{2k}}{1-h} k \|d(y_0, y_1)\|.$

Then, $d(y_{2k}, y_{2m}) \to 0$ as $k, m \to \infty$ so by $\{y_{2k}\} = \{hx_{2k-1}\}$ is a Cauchy sequence.

Since h(X) is a complete subspace of X, so $\exists v \in h(X)$ such that

$\lim_{k \to \infty} y_{2k} = v \quad hx_{2k} \to v$ as $k \to \infty$.

Consequently, we can find in X such that $h(u) = v$.

We shall show that $hu = fu = gu$.

Consider,

$d(fu, y_{2k+1}) = d(fu, gx_{2k+1})$

$$\leq \lambda[d(fu, hu) + d(gx_{2k+1}, hx_{2k+1}) + d(fu, hx_{2k+1}) + d(hu, gx_{2k+1}) + d(hu, hx_{2k+1})]$$

$$\leq \lambda[d(fu, hu) + d(y_{2k+1}, y_{2k}) + d(fu, y_{2k}) + d(hu, y_{2k+1}) + d(hu, y_m)].$$

Hence, we have

$$\|d(fu, hu)\| \leq k\{\lambda \|d(fu, hu) + d(y_{2k+1}, y_{2k}) + d(fu, y_{2k}) + d(hu, y_{2k+1}) + d(hu, y_m)\|\}$$

If $k \to \infty$, then $\|d(fu, hu)\| = 0$, thus $fu = hu$.

Similarly we can show that $gu = hu$.

Therefore, $fu = gu = hu = v$.

Now we show that f, g, h have a unique point of coincidence.

For this we assume that there exists another point $x^* \in X$ such that,

$fx^* = gx^* = hx^* = y^*$ forsome $x^* \in X$.

Then we have,

$d(y^*, v) = d(fx^*, gu)$

$\leq \lambda[d(fx^*, hx^*) + d(gu, hu) + d(fx^*, hu) + d(gu, hx^*) + d(hx^*, hu)]$

$= \lambda[d(y^*, y^*) + d(v, v) + d(y^*, v) + d(y^*, v) + d(y^*, v)]$

$\leq 3\lambda d(y^*, v)$ which gives a contraction.

Hence $\|d(y^*, v)\| = 0$. so we have $y^* = v$.

Thus the point of coincidence is unique.

If pairs (f, h) and (g, h) are weakly compatible then

$fv = fhu = hfu = hv$

And $\quad gv = ghu = hgu = hv$.

Therefore, $fv = gv = hv = w$ (say). This shows that w is another point of coincidence. Therefore by uniqueness, we must have

. $fv = gv = hv = v$.

Thus is a unique common fixed point of self maps and .

This completes the proof of the theorem.

Theorem 3.2:- Let (X, d) be a cone metric space and P be a normed cone with normal constant K . Suppose the mapping $f, g, h : X \to X$ satisfy the condition

$$d(fx, gy) \leq \alpha d(fx, hx) + \beta[d(fx, hx) + d(gy, hy)] + \gamma[d(fx, hy) + d(gy, hx)] \quad \ldots 3.2.1$$

Where $\alpha + 2\beta + 2\gamma < 1$ is a constant for all $\alpha, \beta, \gamma \in (0,1]$ and $\forall x, y \in X$. If $f(X) \cup g(X) \subseteq h(X)$ and $h(X)$ is a complete subspaceof X . Then the maps have a coincidence point in X . Moreover, if (f, h) and (g, h) are weakly compatible pairs, then f, g & h have a unique common fixed point.

Proof: Let x_0 be an arbitrary point in and we define a sequence $\{y_k\}$ in such that

$y_{2k} = fx_{2k} = hx_{2k+1}$, $\quad k = 0,1,2,3.........$

and $y_{2k+1} = gx_{2k+1} = hx_{2k+2}, k = 0,1,2,3.........$

Consider that,

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$= \alpha d(y_{2k}, y_{2k-1}) + \beta[d(y_{2k}, y_{2k-1}) + d(y_{2k}, y_{2k-1})] + \gamma[d(y_{2k}, y_{2k+1}) + d(y_{2k}, y_{2k-1})]$

$\Rightarrow d(y_{2k}, y_{2k+1}) \leq \dfrac{\alpha+\beta+\gamma}{1-\beta-\gamma} d(y_{2k}, y_{2k-1})$

$\therefore d(y_{2k}, y_{2k+1}) \leq h d(y_{2k}, y_{2k-1})$

Where $h = \dfrac{\alpha+\beta+\gamma}{1-\beta-\gamma} < 1.$

Similarly we can show that

$d(y_{2k+2}, y_{2k+1}) \leq h' d(y_{2k+1}, y_{2k})$ where $h' = \dfrac{\alpha+\beta+\gamma}{1-\beta-\gamma} < 1.$

Therefore, for all $\forall k \in N$ we have,

$d(y_{2k+2}, y_{2k+1}) \leq h d(y_{2k}, y_{2k+1})$

$\leq h^2 d(y_{2k}, y_{2k+1})$
$\leq$ …………………..
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$\leq h^{2k+1} d(y_0, y_1).$

Now for any,

$d(y_{2k}, y_{2m}) \leq d(y_k, y_{2k+1}) + d(y_{2k+1}, y_{2k+2}) + \ldots\ldots\ldots\ldots + d(y_{2m-1}, y_{2m})$
$\leq h^{2k} d(y_0, y_1) + h^{2k+1} d(y_0, y_1) + \ldots\ldots\ldots\ldots + h^{2m-1} d(y_0, y_1)$
$\leq (h^{2k} + h^{2k+1} + h^{2k+2} \ldots\ldots\ldots\ldots) d(y_0, y_1)$
$= \dfrac{h^{2k}}{1-h} d(y_0, y_1).$

Since P is normal with normal constant k.

Therefore, $\left\|d(y_{2k}, y_{2m}\right\| \leq \dfrac{h^{2k}}{1-h} k \left\|d(y_0, y_1)\right\|.$

Then, $d(y_{2k}, y_{2m}) \to 0$ as $k, m \to \infty$ with $0 < \alpha + 2\beta + 2\gamma < 1$

So, $\{y_{2k}\} = \{hx_{2k}\}$ is a Cauchy Sequence.

Now, since h(X) is a complete subspace of, so $\exists v \in h(X)$ such that

$\lim_{n\to\infty} y_{2k} = v \; hx_{2k} \to v$ as $k \to \infty.$

Consequently, we can find in such that $h(u) = v.$

We shall show that $hu = fu = gu.$

For this we consider,

$d(fu, y_{2k+1}) = d(fu, gx_{2k+1})$

$\leq \alpha d(hu, hx_{2k+1}) + \beta[d(fu, hu) + d(gx_{2k+1}, hx_{2k+1})] + \gamma[d(fu, hx_{2k+1}) + d(gx_{2k+1}, hu)]$

$$\leq \alpha d(hu, hx_{2k+1}) + \beta[d(hx_{2k+1}, hu) + d(fu, hx_{2k+1})] + \gamma[d(fu, y_{2k}) + d(y_{2k+1}, hu)].$$

$$\leq \alpha d(hu, hx_{2k+1}) + \beta[d(hx_{2k+1}, hu) + d(gx_{k+1}, fu)] + \gamma[d(fu, y_{2k}) + d(hx_{2k+1}, hu)]$$

$$\Rightarrow d(fu, y_{2k+1}) \leq \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} d(hx_{2k+1}, hu)$$

Since P is a normal cone with normal constant k, so

$$\|d(fu, y_{2k+1})\| = \|d(fu, gx_{2k+1})\| \leq k.\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}\|d(hx_{2k+1}, hu)\| \to 0 \text{ as } k \to \infty$$

Therefore, for larger n we get

$$d(hu, fu) \leq d(v, hx_{2k+2}) \leq d(v, u) = 0$$

Which leads to $d(hu, fu) = 0$ and hence $hu = v = fu$.

Similarly we can show that ,

$$hu = v = fu$$

Therefore we have,

$$v = hu = fu = gu .$$

u is a coincidence point of mappings f, g, h .

Since (f, h) and (g, h) are weakly mapping at point u and contractive condition (3.2.1), we have

$$d(ffu, fu) = d(ffu, gu)$$

$$\leq \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} d(hfu, hu)$$

$$\leq d(hfu, hu)$$

$$= d(fhu, fu)$$

$$= d(ffu, fu)$$

i.e. $d(ffu, fu) \leq d(ffu, fu)$ a contradiction. Therefore

$ffu = fu$. Hence $fu = ffu = fhu = hfu$.

$$\Rightarrow ffu = hfu = fu = v.$$

Therefore $fu(= v)$ is a common fixed point of f and h .

Similarly, we can show that

$$gu = ggu = ghu = hgu$$

$$\Rightarrow ggu = hgu = gu = v.$$

Therefore, $gu = fu(= v)$ is a common fixed point of g and h. Hence by the above discussion we concluded that f, g, h have a common fixed point v. The uniqueness of the common fixed point v follow contractive condition (3.2.1). Indeed let v^* be any other fixed point of f, g, h consider

$d(v, v^*) = d(fv, gv^*)$

$\leq \alpha d(hv, hv^*) + \beta[d(fv, hv) + d(gv^*, hv^*)] + \gamma[d(fx^*, hu) + d(gu, hx^*)]$

$\leq \alpha d(hv, hv^*) + \beta[d(hv^*, hv) + d(fv, hv^*)] + \gamma[d(y^*, v) + d(v, y^*)]$

$\leq \alpha d(hv, hv^*) + \beta[d(hv^*, hv) + d(fv, hv^*)] + \gamma d(y^*, y^*)$

$\leq (\alpha + \beta) d(hv, hv^*) + \beta d(fv, hv^*)$ Which gives a contradiction.

Hence $\|d(v, v^*)\| = 0 \Rightarrow v = v^*$ is a contradiction therefore f, g and h have a unique common fixed point.

Theorem 3.3:- Let (X, d) be a cone metric space and P be a normed cone with normal constant K .Suppose the mapping $f, g, h : X \to X$ satisfy the condition

$$d(fx, gy) \leq \alpha[d(fx, hx) + d(gy, hy)] + \beta[d(hx, hy) + d(fx, hy) + d(gy, hx)] \quad \text{………..(3.3.1)}$$

Where $\alpha + 2\beta < 1$ is a constant for all $\alpha, \beta \in (0,1]$ and $\forall x, y \in X$. If $f(X) \cup g(X) \subseteq h(X)$ and $h(X)$ is a complete subspace of X . Then the maps have a coincidence point in X . Moreover,if (f, h) and (g, h) are weakly compatible pairs, then f, g & h have a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X and we define a sequence $\{y_k\}$ in X such that

$y_{2k} = fx_{2k} = hx_{2k+1}$ $\qquad k = 0,1,2,3$……….

and $y_{2k+1} = gx_{2k+1} = hx_{2k+2}$ $\qquad k = 0,1,2,3$……….

Consider that from (3.3.1) we have,

$d(y_{2k}, y_{2k+1}) = d(fx_{2k}, gx_{2k+1})$

$\leq \alpha[d(fx_{2k}, hx_{2k}) + d(gx_{2k+1}, hx_{2k+1})] + \beta[d(hx_{2k}, hx_{2k+1}) + d(fx_{2k}, hx_{2k+1}) + d(gx_{2k+1}, hx_{2k})]$

$= \alpha[d(fx_{2k}, gx_{2k-1}) + d(gx_{2k+1}, fx_{2k})] + \beta[d(fx_{2k}, gx_{2k-1}) + d(y_{2k}, y_{2k+1}) + d(y_{2k}, y_{2k-1})]$

$\leq \alpha[d(y_{2k}, y_{2k-1}) + d(y_{2k+1}, y_{2k})] + \beta[d(y_{2k}, y_{2k-1}) + d(y_{2k}, y_{2k+1}) + d(y_{2k}, y_{2k-1})]$

$\leq \alpha[d(y_{2k}, y_{2k-1}) + d(y_{2k+1}, y_{2k})] + \beta[2d(y_{2k}, y_{2k-1}) + d(y_{2k}, y_{2k+1})]$

$$\Rightarrow d(y_{2k}, y_{2k+1}) \leq \frac{\alpha + 2\beta}{1 - \alpha - \beta} d(y_{2k}, y_{2k-1})$$

$\therefore d(y_{2k}, y_{2k+1}) \leq h d(y_{2k}, y_{2k-1})$

Where $h = \dfrac{\alpha + 2\beta}{1 - \alpha - \beta} < 1$.

Similarly we can show that

$d(y_{2k+2}, y_{2k+1}) \le h' d(y_{2k+1}, y_{2k})$ where $h' = \dfrac{\alpha + 2\beta}{1-\alpha-\beta} < 1$.

Therefore, for all $\forall k \in N$ we have,

$d(y_{2k+2}, y_{2k+1}) \le h d(y_{2k}, y_{2k+1})$

$\le h^2 d(y_{2k}, y_{2k+1})$
$\le$ ……………………..
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$\le h^{2k+1} d(y_0, y_1)$.

Now for any m>k,

$d(y_{2k}, y_{2m}) \le d(y_{2k}, y_{2k+1}) + d(y_{2k+1}, y_{2k+2}) + \ldots\ldots\ldots\ldots + d(y_{2m-1}, y_{2m})$
$\le h^{2k} d(y_0, y_1) + h^{2k+1} d(y_0, y_1) + \ldots\ldots\ldots\ldots + h^{2m-1} d(y_0, y_1)$
$\le (h^{2k} + h^{2k+1} + h^{2k+2} \ldots\ldots\ldots\ldots) d(y_0, y_1)$
$= \dfrac{h^{2k}}{1-h} d(y_0, y_1)$.

Since P is normal with normal constant k.

Therefore, $\|d(y_{2k}, y_{2m})\| \le \dfrac{h^{2k}}{1-h} k \|d(y_0, y_1)\|$.

Then, $d(y_{2k}, y_{2m}) \to 0$ as $k, m \to \infty$ with $0 < \alpha + 2\beta < 1$.

So $\{y_{2k}\} = \{hx_{2k}\}$ is a Cauchy sequence.

Now, Sinceis a complete subspace of so $\exists v \in h(X)$ such that

$\lim_{n \to \infty} y_{2k} = v$ $hx_{2k} \to v$ as $k \to \infty$.

Consequently, we can find in such that $h(u) = v$.

We shall show that $hu = fu = gu$.

For this we consider,

$d(fu, y_{2k+1}) = d(fu, gx_{2k+1})$

$\le \alpha[d(fu, hu) + d(gx_{2k+1}, hx_{2k+1})] + \beta[d(hu, hx_{2k+1}) + d(fu, hx_{2k+1}) + d(gx_{2k+1}, hu)]$

$\le \alpha[d(hx_{2k+1}, hu) + d(fu, hx_{2k+1})] + \beta[d(hu, hx_{2k+1}) + d(fu, y_{2k}) + d(y_{2k+1}, hu)]$.
$\le \alpha[d(hx_{2k+1}, hu) + d(gx_{k+1}, fu)] + \beta[d(hu, hx_{2k+1}) + d(fu, y_{2k}) + d(hx_{2k+1}, hu)]$

$\Rightarrow d(fu, y_{2k+1}) \le \dfrac{\alpha + 2\beta}{1-\alpha-\beta} d(hx_{2k+1}, hu)$

Since P is a normal cone with normal constant k, so

$\|d(fu, y_{2k+1})\| = \|d(fu, gx_{2k+1})\| \le k.\dfrac{\alpha + 2\beta}{1-\alpha-\beta} \|d(hx_{2k+1}, hu)\| \to 0$ as $k \to \infty$

Therefore , for larger n we get

$d(hu, fu) \le d(v, hx_{2k+2}) \le d(v, u) = 0$

Implies that $d(hu, fu) = 0$ and hence $hu = v = fu$.

Similarly we can show that

$hu = v = fu$ therefore we have,

$v = hu = fu = gu$.

i.e. u is a coincidence point of mappings f, g, h .

Now we show that have a unique coincidence point.
For this we assume that there exists another point $x^* \in X$ such that,
$fx^* = gx^* = hx^* = y^*$ for some $x^* \in X$.
Then we have,

$$d(y^*, v) = d(fx^*, gu)$$
$$\leq \alpha[d(fu, hu) + d(gx_{2k+1}, hx_{2k+1})] + \beta[d(hu, hx_{2k+1}) + d(fu, hx_{2k+1}) + d(gx_{2k+1}, hu)]$$
$$\leq \alpha[d(y^*, y^*) + d(v, v)] + \beta[d(y^*, hu) + d(y^*, hu) + d(gu, y^*)]$$
$$= 3\beta[d(y^*, v) + d(y^*, v) + d(v, y^*)]$$

$\Rightarrow d(y^*, v) \leq 3\beta d(y^*, v)$ which gives a contraction.

Hence $\|d(y^*, v)\| = 0$. so we have $y^* = v$.

Thus the point of coincidence is unique.
If pairs (f, h) and (g, h) are weakly compatible then
$fv = fhu = hfu = hv$

And $gv = ghu = hgu = hv$.

Therefore, $fv = gv = hv = w$ (say). This shows that is another point of coincidence. Therefore by uniqueness, we must have
i.e. $fv = gv = hv = v$.
Thus is a unique common fixed point of self maps and
This completes the proof of the theorem.

4. References :-

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