Some Results Involving Generalized Multivariable Gimel-Function and Gegenbauer Polynomials

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ABSTRACT
In this paper an expension theorem for generalized multivariable Gimel-function has been obtained by using a series on Gegenbauer polynomials due to Askey [1]. This theorem is further utilized to evaluate an integral involving product of multivariable Gimel-function and Gegenbauer polynomials.

KEYWORDS : Multivariable Gimel-function, multiple integral contours, Gegenbauer polynomials, expansion serie, finite integrals.

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1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C},\mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0=\mathbb{N}\cup\{0\}$.

We define a generalized transcendental function of several complex variables.

$$\gimel(z_1,\cdots,z_r)=\gimel^{m_2,n_2;m_3,n_3;\cdots;m_r,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|}\mathrm{z}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\end{array}\right.$$

$$[(\mathrm{a}_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},$$
$$[(\mathrm{b}_{2j};\beta^{(1)}_{2j},\beta^{(2)}_{2j};B_{2j})]_{1,m_2},[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{m_2+1,q_{i_2}},[(b_{3j};\beta^{(1)}_{3j},\beta^{(2)}_{3j},\beta^{(3)}_{3j};B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{m_3+1,q_{i_3}};\cdots;[(\mathrm{b}_{rj};\beta^{(1)}_{rj},\cdots,\beta^{(r)}_{rj};B_{rj})_{1,m_r}],$$

$$[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{n_r+1,p_r}]:\ [(\mathrm{c}^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]$$
$$[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{m_r+1,q_r}]:[(\mathrm{d}^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]$$

$$\left.\begin{array}{l};\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_j)_{n^{(r)}+1,p^{(r)}_i}]\\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_j)_{m^{(r)}+1,q^{(r)}_i}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{m_2}\Gamma^{B_{2j}}(b_{2j}-\sum_{k=1}^{2}\beta^{(k)}_{2j}s_k)\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha^{(k)}_{2j}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha^{(k)}_{2ji_2}s_k)\prod_{j=m_2+1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta^{(k)}_{2ji2}s_k)]}$$

$$\frac{\prod_{j=1}^{m_3}\Gamma^{B_{3j}}(b_{3j}-\sum_{k=1}^{3}\beta_{3j}^{(k)}s_k)\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=m_3+1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{m_r}\Gamma^{B_{rj}}(b_{rj}-\sum_{k=1}^{r}\beta_{rj}^{(k)}s_k)\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=m_r+1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)})]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $m_2,n_2,\cdots,m_r,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2\leqslant q_{i_2},0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant m_r\leqslant q_{i_r},0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kj}^{(l)},B_{kj}\in\mathbb{R}^+;(j=1,\cdots,m_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n_k);(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m_k);(k=1,\cdots,r)$.

$a_{kji_k}\in\mathbb{C};(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r)$.

$b_{kji_k}\in\mathbb{C};(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r)$.

$d_{ji^{(k)}}^{(k)}\in\mathbb{C};(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{C};(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

The contour L_k is in the $s_k(k=1,\cdots,r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k\right)(j=1,\cdots,n_2),\Gamma^{A_{3j}}\left(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k\right)$ $(j=1,\cdots,n_3),\cdots,\Gamma^{A_{rj}}\left(1-a_{rj}+\sum_{i=1}^{r}\alpha_{rj}^{(i)}\right)(j=1,\cdots,n_r)$, $\Gamma^{C_j^{(k)}}\left(1-c_j^{(k)}+\gamma_j^{(k)}s_k\right)(j=1,\cdots,n^{(k)})(k=1,\cdots,r)$to the right of the contour L_k and the poles of $\Gamma^{B_{2j}}\left(b_{2j}-\sum_{k=1}^{2}\beta_{2j}^{(k)}s_k\right)(j=1,\cdots,m_2),\Gamma^{B_{3j}}\left(b_{3j}-\sum_{k=1}^{3}\beta_{3j}^{(k)}s_k\right)(j=1,\cdots,m_3)$ $,\cdots,\Gamma^{B_{rj}}\left(b_{rj}-\sum_{i=1}^{r}\beta_{rj}^{(i)}\right)(j=1,\cdots,m_r)$, $\Gamma^{D_j^{(k)}}\left(d_j^{(k)}-\delta_j^{(k)}s_k\right)(j=1,\cdots,m^{(k)})(k=1,\cdots,r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2}A_i^{(k)}\pi$ where

$$A_i^{(k)}=\sum_{j=1}^{m^{(k)}}D_j^{(k)}\delta_j^{(k)}+\sum_{j=1}^{n^{(k)}}C_j^{(k)}\gamma_j^{(k)}-\tau_{i^{(k)}}\left(\sum_{j=m^{(k)}+1}^{q_i^{(k)}}D_{ji^{(k)}}^{(k)}\delta_{ji^{(k)}}^{(k)}+\sum_{j=n^{(k)}+1}^{p_i^{(k)}}C_{ji^{(k)}}^{(k)}\gamma_{ji^{(k)}}^{(k)}\right)+$$

$$\sum_{j=1}^{n_2}A_{2j}\alpha_{2j}^{(k)}+\sum_{j=1}^{m_2}B_{2j}\beta_{2j}^{(k)}-\tau_{i_2}\left(\sum_{j=n_2+1}^{p_{i_2}}A_{2ji_2}\alpha_{2ji_2}^{(k)}+\sum_{j=m_2+1}^{q_{i_2}}B_{2ji_2}\beta_{2ji_2}^{(k)}\right)+\cdots+$$

$$\sum_{j=1}^{n_r}A_{rj}\alpha_{rj}^{(k)}+\sum_{j=1}^{m_r}B_{rj}\beta_{rj}^{(k)}-\tau_{i_r}\left(\sum_{j=n_r+1}^{p_{i_r}}A_{rji_r}\alpha_{rji_r}^{(k)}+\sum_{j=m_r+1}^{q_{i_r}}B_{rji_r}\beta_{rji_r}^{(k)}\right) \quad (1.4)$$

Following the lines of Braaksma ([3] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,),max(\,|z_1|,\cdots,|z_r|\,)\rightarrow 0$

$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,),min(\,|z_1|,\cdots,|z_r|\,)\rightarrow\infty$ where $i=1,\cdots,r$:

$\alpha_i=\min_{\substack{1\leqslant j\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)$ and $\beta_i=\max_{\substack{1\leqslant j\leqslant n_i\\1\leqslant j\leqslant n^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}A_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}}+C_j^{(i)}\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)$

Remark 1.
If $m_2=n_2=\cdots=m_{r-1}=n_{r-1}=p_{i_2}=q_{i_2}=\cdots=p_{i_{r-1}}=q_{i_{r-1}}=0$ and $A_{2j}=B_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=$ $A_{rj}=B_{rj}=A_{rji_r}=B_{rji_r}=1$, then the generalized multivariable Gimel-function reduces in the generalized multivariable Aleph- function (extension of multivariable Aleph-function defined by Ayant [2]).

Remark 2.
If $m_2=n_2=\cdots=m_r=n_r=p_{i_2}=q_{i_2}=\cdots=p_{i_r}=q_{i_r}=0$ and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2=$ $=\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then the generalized multivariable Gimel-function reduces in a generalized multivariable I-function (extension of multivariable I-function defined by Prathima et al. [5]).

Remark 3.
If$A_{2j}=B_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=B_{rj}=A_{rji_r}=B_{rji_r}=1$and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2$ $=\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then the generalized multivariable Gimel-function reduces in generalized of multivariable I-function (extension of multivariable I-function defined by Prasad [4]).

Remark 4.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the generalized multivariable H-function (extension of multivariable H-function defined by Srivastava and panda [7,8]).

In your investigation, we shall use the following notations.

$$\mathbb{A} = [(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.5}$$

$$\mathbf{A} = [(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A = [(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,n^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_j^{(r)})_{n^{(r)}+1,p_i^{(r)}}] \tag{1.7}$$

$$\mathbb{B} = [(b_{2j};\beta_{2j}^{(1)},\beta_{2j}^{(2)};B_{2j})]_{1,m_2},[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{m_2+1,q_{i_2}},[(b_{3j};\beta_{3j}^{(1)},\beta_{3j}^{(2)},\beta_{3j}^{(3)};B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{m_3+1,q_{i_3}};\cdots;[(\mathrm{b}_{(r-1)j};\beta_{(r-1)j}^{(1)},\cdots,\beta_{(r-1)j}^{(r-1)};B_{(r-1)j})_{1,m_{r-1}}],$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})_{m_{r-1}+1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B} = [(\mathrm{b}_{rj};\beta_{rj}^{(1)},\cdots,\beta_{rj}^{(r)};B_{rj})_{1,m_r}],[\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{m_r+1,q_{i_r}}] \tag{1.9}$$

$$\mathrm{B} = [(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}];\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_j^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \tag{1.10}$$

$$U = m_2,n_2;m_3,n_3;\cdots;m_{r-1},n_{r-1};V = m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X = p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y = p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

2. Required results.

We require the following relations for your investigation.

Gegenbauer polynomials defined by generating relation ([6], p. 276(1))

Lemma 1.

$$(1-2xt+t^2) = \sum_{n=0}^{\infty} c_n^{\upsilon}(x)t^n \tag{2.1}$$

These polynomials satisfy the following relations which are required in the development of present work.

The orthogonality property ([6], p. 281 (27)-(28))

Lemma 2.

$$\int_{-1}^{1}(1-x^2)^{\upsilon-\frac{1}{2}}C_n^{\upsilon}(x)C_m^{\upsilon}(x)\mathrm{d}x=\left[\begin{array}{c} 0 \text{ if } \mathrm{m}\neq n \\ . \\ \frac{2^{1-2\upsilon}\pi\Gamma(n+2\upsilon)}{n!(\upsilon+n)[\Gamma(\upsilon)]^2} \text{ if m = n}\end{array}\right. \tag{2.2}$$

The expansion as given by Askey [1]

Lemma 3.

$$\sin^{2\beta}\theta C_l^{\beta}(\cos\theta)=\sum_{u=0}^{\infty}A_{u,l}^{\beta,\alpha}C_{l+2u}^{\alpha}\cos(\theta)\sin^{2\alpha}\theta \tag{2.3}$$

where $\frac{\alpha-1}{2}<\beta<\alpha, A_{u,l}^{\beta,\alpha}>0$ and

$$A_{u,l}^{\beta,\alpha}=\frac{4^{\alpha-\beta}\Gamma(\alpha)(l+2u+\alpha)(l+2u)!\Gamma(l+2\beta)\Gamma(l+u+\alpha)\Gamma(u+\alpha+\beta)}{l!u!\Gamma(\beta)\Gamma(\alpha-\beta)\Gamma(l+u+\beta+1)\Gamma(l+2u+2\alpha)} \tag{2.4}$$

The expansion formula ([6], p. 283 (37))

Lemma 4.

$$C_n^{\upsilon}(\cos\theta)=\sum_{k=0}^{n}\frac{(-n)_k(\upsilon)_k(\upsilon)_n}{k!n!(1-\upsilon-n)_k}\cos(n-2k)\theta \tag{2.5}$$

3. Expansion formula.

In this section we shall establish the following expansion formula

Theorem 1.

$$\sum_{u=0}^{\infty}\frac{2^{\alpha-2}\Gamma(\alpha)(l+2u+\alpha)(l+2u)!\Gamma(l+u+\alpha)}{u!\Gamma(l+2n+2\alpha)}C_{l+2u}^{\alpha}(\cos\theta)(\sin\theta)^{2\alpha-2}\ \gimel_{X;p_{i_r}+3,q_{i_r}+3,\tau_{i_r}:R_r:Y}^{U;m_r+1,n_r+1:V}$$

$$\left(\begin{array}{c|c} 4^{a_1}z_1 & \mathbb{A};(2-\alpha-u;a_1,\cdots,a_r;1),\mathbf{A},(1;a_1,\cdots,a_r;1),(l+u+2;a_1,\cdots,a_r;1):A \\ \cdot & \cdot \\ \cdot & \cdot \\ 4^{a_r}z_r & \mathbb{B};(l+2;2a_1,\cdots,2a_r;1),\mathbf{B},(2-\alpha;a_1,\cdots,a_r;1):B \end{array}\right)$$

$$=\sum_{k=0}^{l}\frac{(-l)_k}{k!}\cos(l-2k)\theta$$

$$\gimel_{X;p_{i_r}+2,q_{i_r}+3,\tau_{i_r}:R_r:Y}^{U;m_r+2,n_r+1:V}\left(\begin{array}{c|c} \frac{z_1}{\sin^{2a_1}\theta} & \mathbb{A};(1+l;a_1,\cdots,a_r;1),\mathbf{A},(1;a_1,\cdots,a_r;2),A \\ \cdot & \cdot \\ \cdot & \cdot \\ \frac{z_r}{\sin^{2a_r}\theta} & \mathbb{B};(1+k;a_1,\cdots,a_r;1),(1+l;a_1,\cdots,a_r;1),\mathbf{B},(1+l-k;a_1,\cdots,a_r;1):B \end{array}\right) \tag{3.1}$$

provided

$$0<\theta<\pi,a_i>0(i=1,\cdots,r),\quad Re(\alpha+u-1)+\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant j\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0.$$

$$Re(-l)-\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant j\leqslant n_i\\1\leqslant j\leqslant n^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}A_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}}+C_j^{(i)}\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)>0 \quad \text{and}$$

$|arg\,(4^{a_i}z_i)| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Proof

Expressing the generalized multivariable Gimel-function on the left sand side of (3.1) as Mellin-Barnes multiple integrals contour with the the help of (1.1), interchanging the order of summation and integration which is justified under the conditions mentioned above, we get

$$\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{i=1}^r\theta_i(s_i)z_i^{s_i}\sum_{u=0}^{\infty}\frac{2^{\alpha+2\sum_{i=1}^r a_is_i-2}\Gamma(\alpha)(l+2u+\alpha)(l+2u)!\Gamma(l+u+\alpha)}{u!l!\Gamma(1-\sum_{i=1}^r a_is_i)\Gamma(\alpha-1-\sum_{i=1}^r a_is_i)}$$

$$C_{l+2u}^{\alpha}(\cos\theta)(\sin\theta)^{2\alpha-2}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.2}$$

Evaluating the series inside the multiple integrals contour with the help of lemma 3, we obtain

$$\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{i=1}^r\theta_i(s_i)z_i^{s_i}\,(\sin\theta)^{2-2\sum_{i=1}^r a_is_i}C_l^{1-\sum_{i=1}^r a_is_i}(\cos\theta)\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.3}$$

Now using the lemma 4 and interchanging the order of summation and integration, we get

$$\sin^2\theta\sum_{k=0}^{l}\frac{(-l)_k}{k!}\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{i=1}^r\theta_i(s_i)\frac{\Gamma(1+k-\sum_{i=1}^r a_is_i)\Gamma(1+l-\sum_{i=1}^r a_is_i)}{\Gamma(1-\sum_{i=1}^r a_is_i)^2}$$

$$\frac{\Gamma(1-l+\sum_{i=1}^r a_is_i)}{\Gamma(1-l+k+\sum_{i=1}^r a_is_i)}\prod_{i=1}^r\left(\frac{z_i}{\sin^{2a_i}\theta}\right)^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.4}$$

and Interpreting the resulting expression with the help of (1.1), we obtain the desired theorem 1.

4. Main integral.

The following integral will be evaluated in this section.

Theorem 2.

$$\sum_{k=0}^{l}\frac{(-l)_k}{k!}\int_0^{\pi}\sin^2\theta C_{l+2u}^{\alpha}(\cos)\cos(l-2k)\theta$$

$$\gimel_{X;p_{i_r}+2,q_{i_r}+3,\tau_{i_r}:R_r:Y}^{U;m_r+2,n_r+1:V}\left(\begin{array}{c|c}\frac{z_1}{\sin^{2a_1}\theta} & \mathbb{A};(1+l;a_1,\cdots,a_r;1),\mathbf{A},(1;a_1,\cdots,a_r;2),A\\ \cdot & \cdot\\ \cdot & \cdot\\ \frac{z_r}{\sin^{2a_r}\theta} & \mathbb{B};(1+k;a_1,\cdots,a_r;1),(1+l;a_1,\cdots,a_r;1),\mathbf{B},(1+l-k;a_1,\cdots,a_r;1):B\end{array}\right)\mathrm{d}\theta$$

$$=\frac{\pi\Gamma(l+u+\alpha)}{2u!\Gamma(\alpha)}\gimel_{X;p_{i_r}+3,q_{i_r}+3,\tau_{i_r}:R_r:Y}^{U;m_r+1,n_r+1:V}$$

$$\left(\begin{array}{c|c}4^{a_1}z_1 & \mathbb{A};(2-\alpha-u;a_1,\cdots,a_r;1),\mathbf{A},(1;a_1,\cdots,a_r;1),(l+u+2;a_1,\cdots,a_r;1):A\\ \cdot & \cdot\\ \cdot & \cdot\\ 4^{a_r}z_r & \mathbb{B};(l+2;2a_1,\cdots,2a_r;1),\mathbf{B},(2-\alpha;a_1,\cdots,a_r;1):B\end{array}\right) \tag{3.5}$$

under the same existence conditions that theorem 1.

Proof

Multiplying both sides of (3.1) by $C_{l+2v}^{\alpha}(\cos\theta)$, integrating with respect to θ from 0 to π, on the left hand side interchanging the order of summation and integration and using the orthogonality property of Gegenbauer polynomials with the help of lemma 2, we get the formula (3.5).

4. Special cases.

In this section we discuss two interesting particular cases of the theorem 1. Taking $\alpha = 1$ and using the following relation $C_n^1(\cos\theta) = \sin(n+1)\theta(\sin\theta)^{-1}$, we have

Corollary 1.

$$\sum_{n=0}^{\infty} \frac{(l+u)!}{u!} \sin(l+2u+1)\theta \, \daleth_{X;p_{i_r}+1,q_{i_r}+2,\tau_{i_r}:R_r:Y}^{U;m_r+1,n_r+1:V}$$

$$\left(\begin{array}{c|c} 4^{a_1}z_1 & \mathbb{A}; (2-u; a_1,\cdots,a_r;1), \mathbf{A}, (1;a_1,\cdots,a_r;1), (l+u+2;a_1,\cdots,a_r;1) : A \\ \cdot & \cdot \\ \cdot & \cdot \\ 4^{a_r}z_r & \mathbb{B}; (l+2; 2a_1,\cdots,2a_r;1), \mathbf{B}, (1,\cdots,a_r;1) : B \end{array} \right) =$$

$$\sin\theta \sum_{k=0}^{l} \frac{(-l)_k}{k!} \cos(l-2k)\theta$$

$$\daleth_{X;p_{i_r}+2,q_{i_r}+3,\tau_{i_r}:R_r:Y}^{U;m_r+2,n_r+1:V} \left(\begin{array}{c|c} \frac{z_1}{\sin^{2a_1}\theta} & \mathbb{A}; (1+l; a_1,\cdots,a_r;1), \mathbf{A}, (1;a_1,\cdots,a_r;2), A \\ \cdot & \cdot \\ \cdot & \cdot \\ \frac{z_r}{\sin^{2a_r}\theta} & \mathbb{B}; (1+k;a_1,\cdots,a_r;1), (1+l;a_1,\cdots,a_r;1), \mathbf{B}, (1+l+k;a_1,\cdots,a_r;1) : B \end{array} \right) \tag{4.1}$$

provided

$$0<\theta<\pi, a_i>0 (i=1,\cdots,r), \quad Re(u) + \sum_{i=1}^{r} a_i \min_{\substack{1\leqslant j\leqslant m_i \\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}} + D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) > 0.$$

$$Re(-l) - \sum_{i=1}^{r} a_i \min_{\substack{1\leqslant j\leqslant n_i \\ 1\leqslant j\leqslant n^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} A_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}} + C_j^{(i)}\frac{c_j^{(i)}-1}{\gamma_j^{(i)}} \right) > 0 \quad \text{and}$$

$|arg\,(4^{a_i}z_i)| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

In the above equation, taking $l=0$, we get an interesting formula :

Corollary 2.

$$\sum_{n=0}^{\infty} \sin(2u+1)\theta \, \daleth_{X;p_{i_r}+3,q_{i_r}+2,\tau_{i_r}:R_r:Y}^{U;m_r+1,n_r+1:V} \left(\begin{array}{c|c} 4^{a_1}z_1 & \mathbb{A}; (1-u;a_1,\cdots,a_r;1), \mathbf{A}, (1;a_1,\cdots,a_r;1), (u+2;a_1,\cdots,a_r;1) : A \\ \cdot & \cdot \\ \cdot & \cdot \\ 4^{a_r}z_r & \mathbb{B}; (2;2a_1,\cdots,2a_r;1), \mathbf{B}, (1,\cdots,a_r;1) : B \end{array} \right)$$

$$= \sin\theta \, \daleth_{X;p_{i_r},q_{i_r},\tau_{i_r}:R_r:Y}^{U;m_r,n_r:V} \left(z_1(\sin\theta)^{-2a_1}, \cdots, z_r(\sin\theta)^{-2a_r} \right) \tag{4.2}$$

provided

$$0<\theta<\pi, a_i>0 (i=1,\cdots,r), \quad Re(u) + \sum_{i=1}^{r} a_i \min_{\substack{1\leqslant j\leqslant m_i \\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}} + D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) > 0.$$

$$-\sum_{i=1}^{r} a_i \min_{\substack{1\leqslant j\leqslant n_i \\ 1\leqslant j\leqslant n^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} A_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}} + C_j^{(i)}\frac{c_j^{(i)}-1}{\gamma_j^{(i)}} \right) > 0 \quad \text{and}$$

$|arg\,(4^{a_i} z_i)| < \frac{1}{2} A_i^{(k)} \pi$ where $A_i^{(k)}$ is defined by (1.4).

5. Conclusion.

The importance of our all the results lies in their manifold generality. By specialising the various parameters as well as variables in the multivariable Gimel-function, we get a several Gegenbauer expansion formulae involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several intersting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

REFERENCES.

[1] R. Askey, Orthogonality expansions with positive coefficients, Proc. Amer. Math. Soc. 16(1965), 1191-1194.

[2] F. Ayant, An integral associated with the Aleph-functions of several variables. International Journal of Mathematics Trends and Technology (IJMTT), 31(3) (2016), 142-154.

[3] B.L.J. Braaksma, Asymptotics expansions and analytic continuations for a class of Barnes-integrals, Compositio Math. 15 (1962-1964), 239-341.

[4] Y.N. Prasad, Multivariable I-function , Vijnana Parishad Anusandhan Patrika 29 (1986) , 231-237.

[5] J. Prathima, V. Nambisan and S.K. Kurumujji, A Study of I-function of Several Complex Variables, International Journal of Engineering Mathematics Vol (2014), 1-12.

[6] E.D. Rainville, Special functions, MacMillan and Co, New York, 1960.

[7] H.M. Srivastava and R. Panda, Some expansion theorems and generating relations for the H-function of several complex variables. Comment. Math. Univ. St. Paul. 24 (1975),119-137.

[8] H.M. Srivastava and R. Panda, Some expansion theorems and generating relations for the H-function of several complex variables II. Comment. Math. Univ. St. Paul. 25 (1976), 167-197.