(3,2)-Jection Operator

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Abstract: *Our main objective in the present paper is to provide some appropriate tools which are necessary for decomposing a (3,2)-jection operator that is not a projection.*

Key words : *Linear operator, Projection operator, (3,2)-jection operator, Orthogonal linear operators, Commuting operators.*

Introduction :

In this paper, we find results concerning with a (3,2)-jection operator so that a (3,2)-jection operator which is not a projection can be decomposed into two orthogonal (3,2)-jections and how expressed as a sum of two commuting operators whose product is 0 where one of the two being a projection and the other whose square is 0.

Definitions :

Linear operator : The operator E on a linear space L is said to be a linear operator if $E(ax+by)=aE(x)+bE(y)$ for all $x, y \in L$ and for scalars a and b.

Projection operator : The operator E on a linear space L is a projection on some subspace M of L if $E^2=E$.

(3, 2)-jecion operator : The operator E on a linear space L is said to be a (3, 2)-jection operator if $E^3 = E^2$.

Commuting operators : Two operators E_1 and E_2 are said to be commuting operators if $E_1E_2=E_2E_1$.

Orthogonal operators : The two operators E_1&E_2 are said to be orthogonal to each other if $E_1E_2 = 0 = E_2E_1$

Main Results :

Theorem-1 :

Any (3, 2)-jection, which is not a projection can be expressed as a sum of two commuting operators whose product is O, one of them being a projection and the other whose square is 0 and conversely.

Proof : Let E be a (3, 2)-jection then $E^3=E^2$

Now we can have,

$E=E-E^2+E^2$

$E=E_1+F_1$

where $E_1=E-E^2$, $F_1=E^2$

Since E is not a projection

$$\therefore E \neq E^2 \Rightarrow E - E^2 \neq 0$$

$$\Rightarrow E_1 \neq 0$$

Here $$E_1F_1 = (E - E^2)E^2 = E^3 - E^4 = E^2 - E^2 = 0$$

& $$E_1^2 = (E - E^2)^2 = E^2 + E^4 - 2E^3 = E^2 + E^2 - 2E^2 = 0$$

$$\& \ F_1^2 = (E^2)^2 = E^4 = E^2 = F_1$$

Thus F_1 is a projection and square of E_1 is 0

Obviously, E_1 and F_1 commute because

$$E_1F_1 = (E - E^2)E^2 = E^3 - E^4 = E^2 - E^2 = 0$$

$$\& \ F_1E_1 = E^2(E - E^2) = E^3 - E^4 = E^2 - E^2 = 0$$

$$\text{i.e., } E_1F_1 = F_1E_1 = 0$$

Conversely,

Let $E = E_1 + F_1$ such that $E_1F_1 = 0, E_1^2 = E_1, F_1 = 0$

Then, $E^2 = (E_1 + F_1)^2$

$$= (E_1 + F_1)(E_1 + F_1)$$

$$= (E_1 + F_1)E_1 + (E_1 + F_1)F_1$$

$$= E_1^2 + F_1E_1 + E_1F_1 + F_1^2$$

$$= E_1 + 0 + 0 + 0$$

$$= E_1$$

$$\text{and}, E^3 = (E_1 + F_1)^3$$

$$= (E_1 + F_1)^2 (E_1 + F_1)$$

$$= E_1(E_1 + F_1)$$

$$= E_1^2 + E_1F_1$$

$$= E_1 + 0 = E_1$$

Thus E is a (3, 2)-jection

Theorem-2 :

Every (3, 2)-jection not a projection is a sum of two mutually orthogonal (3, 2)-jections both of them also not being projections and conversely.

Proof : Let E be a (3, 2)-jection, then $E^3 = E^2$ & $E^2 \neq E$

We can have,

$$E = \frac{1}{2}(E + E^2) + \frac{1}{2}(E - E^2)$$

$$= P + Q \text{ (say) where } P = \frac{1}{2}(E + E^2) \;\&\; Q = \frac{1}{2}(E - E^2)$$

We observe that,

$$P^2 = \left\{\frac{1}{2}(E + E^2)\right\}^2 = \frac{1}{4}(E + E^2)^2 = \frac{1}{4}(E^2 + E^4 + 2EE^2)$$

$$= \frac{1}{4}(E^2 + E^2 + 2E^2) = E^2 \neq P$$

$$\& \; P^3 = P^2P = E^2\frac{1}{2}(E + E^2) = \frac{1}{2}(E^3 + E^4)$$

$$= \frac{1}{2}(E^2 + E^2) = E^2$$

Hence P is a (3, 2)-jection but not a projection.

Next, $Q^2 = \left\{\frac{1}{2}(E - E^2)\right\}^2 = \frac{1}{4}(E^2 + E^4 - 2E^3) = \frac{1}{4}(E^2 + E^2 - 2E^2) = 0 \neq Q$

$$\& \; Q^3 = Q^2Q = 0.Q = 0$$

$\Rightarrow$ Q is a (3, 2)-jections which is not a projection.

Again, $PQ = \frac{1}{2}(E + E^2).\frac{1}{2}(E - E^2) = \frac{1}{4}(E^2 - E^4) = \frac{1}{4}(E^2 - E^2) = 0$

$\Rightarrow$ P & Q are orthogonal.

Hence E has been expressed as a sum of two mutually orthogonal (3, 2)-jections where both of the two are not projections.

Conversely,

Let $E = E_1 + F_1$ where E_1& F_1 are (3, 2)-jections such that

$E_1F_1 = 0 = F_1E_1$

& $E_1^2 \neq E_1, F_1^2 \neq F_1$

We have,

$$
\begin{aligned}
E^2 = (E_1 + F_1)^2 &= E_1^2 + F_1^2 + 2E_1F_1 \\
&= E_1^2 + F_1^2 + 2(0) \\
&= E_1^2 + F_1^2
\end{aligned}
$$

$$
\begin{aligned}
\& \ E^3 = E^2E = (E_1^2 + F_1^2)(E_1 + F_1) &= E_1^3 + F_1^3 + E_1^2F_1 + F_1^2E_1 \\
&= E_1^3 + F_1^3 + E_1(E_1F_1) + F_1(F_1E_1) \\
&= E_1^3 + F_1^3 + E_1(0) + F_1(0) \\
&= E_1^3 + F_1^3 \\
&= E_1^2 + F_1^2
\end{aligned}
$$

i.e. $E^2 = E^3$

$\Rightarrow$ E is a (3, 2)-jection.

Conclusion :

A (3,2)-jection operator not a projection can be decomposed into two orthogonal (3,2)-jections and also be expressed as a sum of two commuting operators whose product is 0 where one of the two being a projection and the other whose square is 0.

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