On Pseudo-Differential Algebraic Functions

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Abstract

This paper develops pseudo-differential algebraic functions which unequivocally incorporate most typical conventional systems and problems of the parabolic limit value discerned forms. Therefore, to achieve this, this paper develops a pseudo-differential parabolic operator's theory in anisotropic spaces. A significant calculus is developed for various classical symbols which are defined universally by $\mathbb{R}^{n+1} * \mathbb{R}^{n+1}$. A periodical procedure regarding the symbolic calculus in a cylinder such that $T^n * \mathbb{R}$ is developed. The Garding's inequality is exhibited for its appropriate operators as well as definite estimates for the vital criterion of the Sobolev anisotropic spaces.

Introduction

This paper is motivated by the behavior of various boundary or limit value element sequence from the heat equation. Pseudo-differential operator analysis hasbeenhighly appreciatedfor the elliptical limit value problems. However, researchers have not fully exploited the pseudo-differential operators formed by the reduction of the limit value of the parabolic functions. Therefore, the Garding's inequality and the estimates of the vital criterion in Sobolev anisotropic spaces for limitation techniques are yet to be researched. Only several parts of the rationale of accustomed pseudo-differential parabolic operands exist. For instance, Piriou (1970) came up with a rationale for parabolic limit value problems as well as pseudo-differential parabolic operands as represented symbolically through the use of expanded functions in quasi-homogeneous functions. Earlier work by Hunt and Piriou (1969) prepared the results of the expanded quasi-homogeneous functions which extended to the pseudo-differential operator's calculus.

This study develops a given pseudo-differential operator's calculus containing symbolic anisotropies having the Piriou's limit integral operand to express the parabolic limit value problems. This study uses a class of operands which is well explained in Beals' (1975) study on the pseudo-differential operators general calculus. In his work, Beals clearly shows the relationship between L^2 and continuity of the Sobolev spaces applying various theorems. However, this study will opt to apply the fundamental method as elaborated by Hormander (2013). This fundamental method refrains from the theory of perfectly sustained operators and uses the calculus of symbols which are universally defined. It is made fundamental by the fact of obtaining the continuity of L^2 through basic properties of infirm unique integral operands. The concept used in this study presents essential tools for administering periodization to satisfy the need for coverage of cylindrical realm.

The Garding's inequality also is proved as well as the distinct criterion measures. The results of Noon (1988) and Costabel (1990) are achieved for the elementary parabolic limit integral operand.

Definitions

If $m \in \mathbb{R}$ such that $m \geq 1$, then ρ, which is the anisotropic span within $\mathbb{R}^{n+1}$ given (Costabel&Saranen, 2001);

$$\rho(\zeta) = \rho_m(\zeta) = |\eta|^{\frac{1}{m}} + |\xi| \text{This is for } \zeta = (\xi, \eta), \quad \xi \epsilon \mathbb{R}, \quad \eta \epsilon \mathbb{R} \quad (i)$$

To define the classical symbol S_m^{β}, then we follow;

$z = (x,t)\epsilon\mathbb{R}^{n+1}$ while $v = (v^1, v^2)\epsilon\mathbb{N}_0^n * \mathbb{N}_0, \partial_z^v = \partial_x^{v^1}\partial_t^{v^2}$, Also $\partial_\zeta^\mu = \partial_\xi^{\mu^1}\partial_\eta^{\mu^2}$ considering $\mu = (\mu^1, \mu^2)\epsilon\mathbb{N}_0^n * \mathbb{N}_0$. Furthermore, from $|v| = |v^1| + v^2$ such that $|v^1| = v_1^1 + \cdots + v_n^1$ considering $v^1 = (v_1^1 \ldots, v_n^1)$, so that we obtain $|\mu|_m = |\mu^1| + m\mu^2$

Definition 1

If $m \in \mathbb{R}$ and $\beta\epsilon\mathbb{R}$, such that $m \geq 1$. Then $a\epsilon C^\infty(\mathbb{R}_z^{n+1} * \mathbb{R}_\zeta^{n+1})$ becomes S_m^β if $v = (v^1, v^2)\epsilon\mathbb{N}_0^{n+1}$ and $\mu = (\mu^1, \mu^2)\epsilon\mathbb{N}_0^{n+1}$ then a constant $C_{v,\mu}$ exists so that(Costabel&Saranen, 2001),

$$\left|\partial_z^v\partial_\zeta^\mu a(z,\zeta)\right| \leq C_{v,\mu}\left(1 + \rho(\zeta)\right)^{\beta - |\mu|m} for\ \forall z, \zeta\epsilon\mathbb{R}^{n+1} \qquad (ii)$$

Having $a(z,\zeta)\epsilon S_m^\beta$ the pseudo-differential operand $a(z,D)$ is expressed as;

$$a(z,D)u(z) = (2\pi)^{-(n+1)}\int e^{i(z,\zeta)}\,a(z,\zeta)\hat{u}(\zeta)d(\zeta) \qquad (iii)$$

The Schwartz space that indicates fast reducing functions is expressed by $S(\mathbb{R}^{n+1})$ while the double space is expressed by $S'(\mathbb{R}^{n+1})$. The Fourier transform is denoted by $\hat{u}\epsilon S(\mathbb{R}^{n+1})$, its expressed specifically by;

$$\hat{u}(\zeta) = \int e^{-i(z,\zeta)}\,a(z,\zeta)u(z)d(z)$$

The scalar yield of $\mathbb{R}^{n+1}$ is denoted as (z,ζ) which forms a bilinear expression. Supposing that, $a(z,\zeta) = \zeta_j\,(j = 1, \ldots\ldots, n+1)$, then we can say that $a(z,D) = D_j = -i\frac{\partial}{\partial z_j}$. For equation (iii) above, the defined operators $a(z,D) \in OpS_m^\beta$.can be symbolically written as $a \in S_m^\beta$.

Beals (1975) showed that the relationship between pseudo-differential analysis of symbolic anisotropies with the theory of weight vectors. For instance, if we express the vector of weight $(\varphi_1, \ldots \varphi_2, \ldots \varphi_{n+1}, \phi_1, \ldots \phi_2, \ldots \phi_{n+1})$ through $\varphi_j(z,\zeta) = 1 (j = 1, \ldots, n+1)$ with $\phi_j(z,\zeta) = 1 + \rho(\zeta)(j = 1, \ldots, n+1)$ and $\phi_{n+1}(z,\zeta) = (1 + \rho(\zeta))^m$.

The selective property in the anisotropic span ρ, which also indicates the conditions required to satisfy the principle of the vectors of weight (Beals, 1975), can be expressed for the triangular disparity as,

$$\rho(\zeta' + \zeta) \leq \rho(\zeta') + \rho(\zeta) \qquad for\ \ (\zeta', \zeta\epsilon\mathbb{R}^{n+1})\ \ for\ every\ m \geq 1 \qquad (iv).$$

It is, however, crucial to understand that the balls under anisotropic defined with $\rho(\zeta) < R$ for every m> 1 do not appear convex. The inequality indicated in equation (iv) is crucial for understanding this definition. The symbolic class as defined by S_m^β is a topology scope such that when variables β and m are fixed, then it can be defined using $q_{v,\mu}^\beta, \mu, v\epsilon\mathbb{N}_0^{n+1}$,

$$q_{v,\mu}^\beta(a) = \sup_{z,\zeta}\left|\partial_z^v\partial_\zeta^\mu a(z,\zeta)\left(1 + \rho(\zeta)\right)^{|\mu|_m - \beta}\right| \qquad (v)$$

It can be proved that $a_j\epsilon S_m^\beta$ does converge to $a\epsilon S_m^\beta$ only when;

$$q_{v,\mu}^\beta\left(a_j - a\right) \rightharpoonup 0, \qquad j \longrightarrow \infty \qquad (vi)$$

Again, $\forall v, \mu\epsilon\mathbb{N}_0^{n+1}$as well as $[a_j]$ becomes bounded when;

$$\sup_j q_{v,\mu}^\beta\left(a_j\right) \leq C_{v,\mu} \qquad (vii)$$

For a symbolic $a(z,\zeta)$ a relation of symbols are considered such that $a_\varepsilon(z,\zeta)$ for $\varepsilon \geq 0$. This relation can be expressed by;

$$a_\varepsilon(z,\zeta) = a(z,\zeta_\varepsilon) \qquad such\ that \qquad \zeta_\varepsilon = (\varepsilon\xi,\ \varepsilon^m\eta) \qquad (viii)$$

Definition 2

Considering the function expressed as $u(x,t)$, $z=(x,t)\epsilon\mathbb{R}^{n+1}$ found in the first period of $x_1,\dots,x_n: u(x+k,t)=u(x,t) \quad \forall k\in\mathbb{Z}^n; (x,t)\in\mathbb{R}^{n+1}$

Assuming that u is continuous, it can be said to be a continuous function within $\mathbb{T}^n * \mathbb{R}$ a (Costabel&Saranen, 2001) cylinder in which $\mathbb{T}^n=\left[\frac{\mathbb{R}}{\mathbb{Z}}\right]^n$ forms the torus ranges. It's therefore possible to take $Q^n=[0,1]^n$ so as to represent $\mathbb{T}^n$. In the light of these, we can also take u to be a steady function in $Q^n * \mathbb{R}$ with a recurrent limit value condition in $\partial Q^n * \mathbb{R}$

Taking u to be limited polynomially, then we obtain the following Fourier functions (Costabel&Saranen, 2001).

a) $\hat{u}(\zeta), \zeta\epsilon\mathbb{R}^{n+1}$ is the normal Fourier function according to $u\epsilon S'(\mathbb{R}^{n+1})$. Suppose $u\epsilon S(\mathbb{R}^{n+1})$, then the expression for $\zeta=(\xi,\eta)\epsilon\mathbb{R}^{n+1}$ will be;

$$\hat{u}(\xi,\eta)=\int_{\mathbb{R}_x^n}\int_{\mathbb{R}_t} e^{-i((\xi,x)+t\eta)}u(x,t)dtdx$$

b) $\underline{\hat{u}}\left(\underline{\zeta}\right)$ in $\underline{\zeta}=(k,\eta)\epsilon\mathbb{Z}^n * \mathbb{R}$, expressed as a coefficient of Fourier functions in scope variables;

$$\underline{\hat{u}}(k,\eta)=\int_{Q_x^n}\int_{\mathbb{R}_t} e^{-i(2\pi(k,x)+t\eta)}u(x,t)dtdx$$

This can also be expressed as;

$$\underline{\hat{u}}\left(\underline{\xi}\right)=\int_{\mathbb{T}^n*\mathbb{R}} e^{-i\left(\underline{\zeta}+z\right)}u(z)dz$$

The inverse transform

$$u(z)=\frac{1}{2\pi}\sum_{k\epsilon\mathbb{Z}^n}\int_{\mathbb{R}_\eta} e^{i(2\pi(k,x)+t\eta)}\underline{\hat{u}}(k,\eta)d\eta=\frac{1}{2\pi}\int_{\mathbb{Z}_k^n*\mathbb{R}_\eta} e^{i(\zeta,z)}\underline{\hat{u}}\left(\underline{\zeta}\right)d\underline{\zeta}$$

Therefore, the scalar yield used are;
$(\zeta,z)=(\xi,x)+t\eta$ and $(\underline{\zeta},z)=2\pi(k,x)+t\eta$ used in

$$\zeta=(\xi,\eta)\epsilon\mathbb{R}^{n+1}$$

$\underline{\zeta}=(\xi,\eta)\epsilon\mathbb{T}^n * \mathbb{R}$, and
$z=(x,t)\epsilon\mathbb{R}^{n+1}$.
For ease of use, $k\epsilon\mathbb{Z}^n$ is used to parameterize $(2\pi\mathbb{Z})^n$ which is a set of $\mathbb{T}^n$. Following this, $\mathbb{T}^n * \mathbb{R}$ can now be defined. The symbols in $a(z,\zeta)$ are chosen in $\mathbb{R}^{n+1}$ on the first period as $z_1,\dots z_2$,hence we can express;

$$S_{m,per}^{\beta}=[a(z,\zeta)\epsilon S_m^{\beta}|a(x+k,t;\xi,\eta)=a(x,t;\xi,\eta)\forall k\in\mathbb{Z}^n]$$

Hence, the above equation forms the C^∞factors in $(\mathbb{T}^n * \mathbb{R}) * \mathbb{R}^{n+1}$. Another important aspect is that $S_{m,per}^{\beta}\subset S_m^{\beta}$. Therefore, $a\in S_{m,per}^{\beta}$ in double pseudo-differential operands. The scaled symbol to show this is;

$$a_{2\pi}(x,t;\xi,\eta)=a(x,t;2\pi\xi,\eta) \qquad (ix)$$

We, therefore, define;

i) The function $a(z,D)$ which acts on $\mathbb{R}^{n+1}$ hence for $z\epsilon\mathbb{R}^{n+1}$;

$$a(z,D)u(z)=(2\pi)^{-(n+1)}\int_{\mathbb{R}^{n+1}} e^{i(\zeta,z)}a\,(z,\zeta)\hat{u}(\zeta)d\zeta$$

ii) The Pseudo-differential operand present in $\mathbb{T}^n * \mathbb{R}$, $\underline{a}(z,D)$, is expressed as;

$$\underline{a}(z,D)u(z)=\frac{1}{2\pi}\int_{\mathbb{Z}^n*\mathbb{R}} e^{i(\zeta,z)}a_{2\pi}\,(z,\underline{\zeta})\underline{\hat{u}}\left(\underline{\zeta}\right)d\underline{\zeta}$$

From definition one, $\left(\underline{\zeta},z\right)=2\pi(k,x)+\eta t$ in $\underline{\zeta}=(k,\eta)\epsilon\mathbb{Z}^n * \mathbb{R}$, $z=(t,x)\epsilon\mathbb{T}^n * \mathbb{R}$.

It is crucial to mention that the expression of $\underline{a}(z,D)$ applies the parameters of $a_{2\pi}(z,\zeta)$ in $\underline{\zeta} = (k,\eta)\epsilon\mathbb{Z}^n * \mathbb{R}$. Hence $\underline{a}(z,D)$ expresses the symbol of $a(z,\underline{\zeta})$ in only $(\mathbb{T}^n * \mathbb{R})_z * ((2\pi\mathbb{Z})^n * \mathbb{R})_\zeta$.This is different to $a(z,D)$ which expounds $a(z,\zeta)$ in the whole of $\mathbb{R}^{2(n+1)}$.

Theorems

Theorem 1

Assuming $a(z,\zeta)\epsilon S_m^\beta(\mathbb{R}_z^{n+1} * \mathbb{R}_\zeta^{n+1})$ and also assuming that mapping of η to $a(z,\xi,\eta)$ entails a substantial continuation through the scope of $\eta - i\sigma, \sigma > 0$ in a way that the continuation goes for $\sigma \geq 0$ satisfying (Costabel&Saranen, 2001);

$$|a(z,\xi,\eta - i\sigma)| \leq C\left(1 + |\xi| + |\eta - i\sigma|^{\frac{1}{m}}\right)^\beta, \ for\ \sigma \geq 0 \qquad (x)$$

In this way, the $a(z,D)$ operand is said to be of *Volterra*form.

It can also be expressed in a short form as;

$$V_m^\beta(\mathbb{R}_z^{n+1} * \mathbb{R}_\zeta^{n+1})$$

This mostly applies to those symbols that satisfy the above assumption. To establish the parabolic limit value problems, an initial value is required as well as the limit values. This initial value is also to be used in those limit integral values to solve this problem. In solving this problem, the vanishing initial value is usually considered which is also used in classical pseudo-differential operands of the Volterra form.

Therefore, stating the Sobolev anisotropic scope;

$$\bar{H}_m^s(\mathbb{R}^{n+1}) = \bar{H}_m^s(\mathbb{R}_x^n * \mathbb{R}_t)$$

This scope assumes the vanishing initial terms at *t=0*, therefore;

$$\bar{H}_m^s(\mathbb{R}^{n+1}) = [u\epsilon H_m^s(\mathbb{R}^{n+1}): supp\ u \subset \mathbb{R}_x^n * [0,\infty]]$$

Hence, if considering finite range of time we express it using $\mathbb{R}_T^{n+1} = \mathbb{R}_x^n * (0,T)$ for $T > 0$

$$\bar{H}_m^s(\mathbb{R}_T^{n+1}) = [u = U|_{\mathbb{R}_x^n*(-\infty,T)}: U \in \bar{H}_m^s(\mathbb{R}_T^{n+1})]$$

The concept of $\bar{H}_m^s(\mathbb{R}_T^{n+1})$ is expressed by;

$$\| u \|_{s,T} = \inf[\| U \|_s: u = U|_{\mathbb{R}_x^n*(-\infty,T)}]$$

Theorem 2

Assuming that $a(z,\zeta) \in V_m^\beta(\mathbb{R}_z^{n+1} * \mathbb{R}_\zeta^{n+1})$, and $a(z,D)\ states\ \forall s \in \mathbb{R}$ limited operands(Costabel&Saranen, 2001);

a) $a(z,D): \bar{H}_m^s(\mathbb{R}^{n+1}) \to \bar{H}_m^{s-\beta}(\mathbb{R}^{n+1})$
b) $a(z,D): \bar{H}_m^s(\mathbb{R}_T^{n+1}) \to \bar{H}_m^{s-\beta}(\mathbb{R}_T^{n+1})$
c) $\| a(z,D) \|_{L\left(\bar{H}_m^s(\mathbb{R}_T^{n+1}), \bar{H}_m^{s-\beta}(\mathbb{R}_T^{n+1})\right)} \leq \| a(z,D) \|_{L\left(\bar{H}_m^s(\mathbb{R}^{n+1}), \bar{H}_m^{s-\beta}(\mathbb{R}^{n+1})\right)}$

The proof for this is the indication of the mapping boundedness exhibited by part (a).

$$a(z,D): H_m^s(\mathbb{R}^{n+1}) \to H_m^{s-\beta}(\mathbb{R}^{n+1})$$

Together with the Volterra property. In part *(b)* we let the term $u \in \bar{H}_m^s(\mathbb{R}_T^{n+1})$ and then designate $U \in \bar{H}_m^s(\mathbb{R}^{n+1})$ so that $u = U|_{\mathbb{R}_x^n * (-\infty,T)}$. Hence, we can state that;

$$a(z,D)u = a(z,D)U|_{\mathbb{R}_x^n * (-\infty,T)}$$

Therefore from the VolterracharacteristicS the right-hand side becomes independent of U. We can also state that $a(z,D)u \in \bar{H}_m^{s-\beta}(\mathbb{R}_T^{n+1})$ so that the final inequality is;

$$\| a(z,D) \|_{s-\beta;T} = \inf[\| F \|_{s-\beta} : F|_{\mathbb{R}_x^n * (-\infty,T)} = a(z,D)u]$$

$$\leq \inf[\| a(z,D)U \|_{s-\beta} : U|_{\mathbb{R}_x^n * (-\infty,T)} = u]$$

$$\leq \inf[\| a(z,D)U \|_{L\left(\bar{H}_m^s(\mathbb{R}_T^{n+1}), \bar{H}_m^{s-\beta}(\mathbb{R}_T^{n+1})\right)} \| U \|_s : U|_{\mathbb{R}_x^n * (-\infty,T)} = u]$$

$$= \| a(z,D) \|_{L\left(\bar{H}_m^s(\mathbb{R}_T^{n+1}), \bar{H}_m^{s-\beta}(\mathbb{R}_T^{n+1})\right)} \| u \|_{s;T}$$

Hence for the above series of theequation, equation (b) and (c) are implied.

From the results obtained in this theorem, the equivalent results are easily obtained for the Sobolev anisotropic scope in $Q_T = \mathbb{T}^n * (0,T)$ which is a finite cylinder. Therefore the scope defined by $\bar{H}_m^s(\mathbb{T}^n * \mathbb{R})$ is said to be the scope of functions that disappear for $-t$ and $\bar{H}_m^s(Q_T)$ due to scope restrictions in Q_T.

Proposition

Proposition 1

Taking K to be a poisson operand in the order $m \in R$ such that it was stated as an operand ranging from $C^\infty(\Gamma)$ and extending to $C^\infty(\Omega)$. So as it can map $C^\infty(\bar{S})$ to $C^\infty(\bar{Q})$ at constant t (Grubb &Solonnikov, 1990). Therefore, for all $r \geq \max[m, \frac{1}{2}]$ as well as $s \geq 0$, K stretches to an operand bearing the continuity characteristics;

$$K: H^{r-\frac{1}{2},s}(S) \to H^{r-m,s}(Q) \qquad for\ m \leq \frac{1}{2}$$

$$K: H^{r-\frac{1}{2}\left(r-\frac{1}{2}\right)\frac{s}{r}}(S) \to H^{r-m,\frac{(r-m)s}{r}}(Q) \qquad for\ m \geq \frac{1}{2}$$

This is proved through the knowledge that K is steady such that,

$$K: H^{r-\frac{1}{2}}(\Gamma) \to H^{r-m}(\Omega)$$

Suppose $r \geq m$, including the variable of *t*, the continuity characteristics are indicated as(Grubb &Solonnikov, 1990);

$$K: L^2\left(I; H^{r-\frac{1}{2}}(\Gamma)\right) \to L^2\left(I; H^{r-m}(\Omega)\right),$$

$$K: H^s\left(I; H^{m-\frac{1}{2}}(\Gamma)\right) \to H^s\left(I; L^2(\Omega)\right),$$

Proposition 2

Supposing P is a pseudo-differential operand on R^nwithin the order $m\epsilon\mathbb{Z}$, the transmission characteristic is at Γ; P_Ω is defined along $C^\infty(\overline{\Omega})$ for $P_\Omega = r_\Omega P e_\Omega$ where e_Ω represents a zero extension on $\frac{\mathbb{R}^n}{\Omega}$ while r_Ω is the constraints ranging from $\mathbb{R}^n$ to Ω. Its extended to $C^\infty(\bar{Q})$ in constant *t*. It is such that $\forall r \geq \max[m, 0]$ while all $s \geq 0$, for P_Ω, it extends to an operand having the continuity characteristics (Grubb &Solonnikov, 1990).

$$P_\Omega: H^{r,s}(Q) \rightarrow H^{r-m,s'}(Q) \qquad for\ s' = \min\left[\frac{(r-m)s}{r}, s\right]$$

So the term $\frac{(r-m)s}{r}$ is interpreted as s when r is zero. Also, when S becomes a pseudo-differential operand in the order of $m\epsilon\mathbb{R}$ on the Γ operator. This is then extended to $C^\infty(\bar{S})$ which is a *t*-constant. S is extended to continuity by a steady operator (Grubb &Solonnikov, 1990);

$$S: H^{r,s}(S) \rightarrow H^{r-m,s'}(S)$$

This occurs when *r*, *s* and *s'*are represented as in the above equations

Lemmas

Lemma 1

Assuming $a\epsilon S_m^0$ as well as $0 \leq \epsilon \leq 1$.The classification of a_ε, $0 \leq \varepsilon \leq 1$ is limited in S_m^0 while $a_\varepsilon \rightarrow a_0$ within $S_m^\beta \forall \beta > 0$. It can be more accurately expressed as;

$$\sup_{0 \leq \varepsilon \leq 1} q_{v,\mu}^\beta(a_\varepsilon) \leq C_{v,\mu}$$

And;

$$q_{v,\mu}^\beta(a_\varepsilon - a_0) \leq C_{v,\mu}\varepsilon^{\min[1,\beta]} \qquad for\ \beta > 0$$

Lemma 2

Assuming $a_j\epsilon S_m^{\beta j}$, $\beta j \rightarrow -\infty$. Hence, various symbols exist such as $a\epsilon S_m^{\beta 0}$, so that,

$$a - \sum_{j=0}^{k-1} a_j \in S_m^{\beta k}, \qquad for\ k \in \mathbb{N}$$

Suppose we have $x \in C_0^\infty(\mathbb{R}_\zeta^{n+1})$ as well as $\varepsilon > 0$, $\exists\delta_\varepsilon > 0$ such that

$$|\zeta| \geq \delta_\varepsilon \Rightarrow x(\zeta_\varepsilon) = 0$$

The term$a(z,\zeta)$, is a singular defined modulus which forms as an extra term to $S_m^{-\infty}$. Also, $a(z,\zeta)$ can be applied so that $a \subset \cup_{j=0}^\infty supp\ a_j$. This Lemma can also be expressed in short as;

$$a \sim \sum_{j=0}^\infty a_j$$

Though it is never a requirement for orders to decrease monotonically, orders can be made to adhere so by taking their partial sums and hence assume that for β_j then it follows that $\beta_0 > \beta_1 > \beta_2 > \beta_3 \ldots \ldots$ ..... For such a series the constant a_0 can be said to be the fundamental term of a.

Conclusion

This paper has well developed the algebraic functions of the pseudo-differential forms which entirely contain the fundamental forms of the generalized parabolic limit value problems. The paper has showcased in depth two definitions, two theorems, two prepositions and two lemmas to describe the pseudo-differential algebraic functions fully.

In detail, the pseudo-differential operators have been built in anisotropic spaces and scope. The cylindrical symbolic calculus has been developed using periodical procedures in Theorem 2. The Garding's inequality has also been developed from the definitions and theorems as well as estimates for anisotropic Sobolev spaces and mapping. This paper has focused on most of the unresearched boundary value problems of the generalized forms. To completely understand the nature of pseudo-algebraic functions and the limit value problem, much study effort has to b applied especially regarding the Garding's inequality and the estimates of the anisotropic Sobolev spaces.

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