Fixed Point Theorem Using Generalized Meir-Keeler Contractions on G-Metric Spaces

Anil Rajput[1], Rucha Athaley[2]

[1]Chandra Shekhar Azad P. G. College, Sehore, Madhya Pradesh, India.
[2]Sardar Ajeet Singh Memorial Girls College, Bhopal, Madhya Pradesh, India

Abstract-*In this paper we consider the concept of G-metric space and prove some fixed point theorems for Meir-Keeler co traction. We also present some applications.*
Keywords-*Fixed Point,G-metric space,Meir-Keeler contraction*

1 Introduction and Preliminaries

Fixed point theory has many applications in wide areas of mathematics, also in many branches of quantitative sciencessuch as economics and computer sciences. The Banach contraction principle [13] is the famous result in this field. This result has many generalizations out of which Meir-Keeler contraction is one.

The Meir-Keeler contraction [22] which wasproved in**1969**plays an important role in the field of fixed point theory and it has been extended by many authors.

In **2009,** T.H. Chang and C.M. Chen [14] defined the weaker Meir–Keeler type function $\psi: \Re+\to \Re+$ and proved following common fixed point theorem of two set-valued mappings in a complete metric space:

Let (X, d) be a complete metric space, and let $T, S: X \to B(X)$. If (T, S) have the non -contraction property, and if for each $t > 0$ with $\psi(t) < t$ and $\{\psi n(t)\}n \in N$ is non-increasing, then S and T has a unique common fixed point a in. Moreover,

$Sa = Ta = \{a\}$.

In **2010** C. Chen and T. Chang[15]defined a weaker Meir–Keeler type function $\psi: intP \cup \{0\} \to intP \cup \{0\}$ in a cone metric space, and proved the following common fixed point theorems of four single-valued functions in cone metric spaces:

Let(X, d) be a complete cone metric space with regular cone P such that$d(x, y) \in intP$ forall $x, y \in X$with$x \neq y$, and let $F, G,$ S, T: $X \to$ X be four single-valued functions with $SX \subset GX$ and$TX \subset FX$ such that for all $, y \in X$,

$$d(Sx, Ty) \leq \psi(max\{d(Fx, Gy), d(Fx, Sx), d(Gy, Ty), \frac{1}{2}[d(Fx, Ty) + d(Gy, Sx)]\}).$$

IfS and$F$ are compatible,T and$G$ are compatible, and if either$F$ or$G$ iscontinuous, then$S, T, F$and$G$ have a unique common fixed point inX.
Samet B. in **2010** [39] defined generalized Meir-Keeler type function and proved the following fixed point theorem in partial metric space:
Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space.Let $F : X \times X \to X$ beamappingsatisfyingthefollowinghypotheses:

(i) F iscontinuous,
(ii) F hasthemixedstrictmonotoneproperty,
(iii) FisageneralizedMeir–Keelertypefunction,
(iv)$\exists x_0, y_0 \in X$ suchthat $x_0 < F(x_0, y_0)$and$y_0 \geq F(y_0, x_0)$.
Then,thereexists $(x, y) \in X$ suchthat $x = F(x, y)$and$y = F(y, x)$.

Mustafa and Sims [32]in **2005**introduced G-metric space which was the generalization of metric space.Mustafa proved some fixed point results [24-34, 35, 5, 6] in G-metric spaces. Shatanawi W. [42] also proved some fixed point results in G-metric spaces for ϕ −maps and for two weakly mappings in partially ordered G-metric spaces.

In **2011** Aydi H., Damjanovic [11] proved coupled coincidence and coupled common fixed point theorems for a mixed g -monotone mapping satisfying nonlinear contractions in partially ordered G -metric spaces. Following result was proved by them:

Let$(X, \preceq)$ be a partially ordered set and G be a G -metric on X such that(X, G) is a complete G -metric space. Suppose that there exist$\phi \in \Phi$,$F: X \times X \to X$ and $g: X \to X$ such that
(1) $G(F(x,y), F(u,v), F(w,z)) \le \phi(G(gx, gu, gw) + G(gy, gv, gz))$

(2) for all $x, y, u, v, w, z \in X$ with $gw \preceq gu \preceq gx$ and $gy \preceq gv \preceq gz$.

Suppose also that$F$ is continuous and has the mixedg -monotone property,$F(X \times X) \subseteq g(X)$ and$g$ is continuous and commutes withF. If there exist$x_0, y_0 \in X$ such that

$gx_0 \preceq F(x_0, y_0)$and$F(x_0, y_0) \preceq gy_0$,

then$F$ and$g$ have a coupled coincidence point, that is, there exists$(x, y) \in X \times X$ such that

$gx = F(x, y)$and$gy = F(y, x)$.

In 2012, Aydi H., Karapinar Erdal[9],proved a general common fixed point theorem for two pairs of weakly compatible self-mappings of a partial metric space satisfying a generalized Meir-Keeler type contractive condition.
Let A, B, S and T be any self-maps of a partial metric space(X, p) satisfying the following conditions;

$(C_1) AX \subseteq TX, BX \subseteq SX,$ (1)

(C_2) Given$\epsilon > 0$, there exists a $\delta > 0$ such that for all x, y in X

$\varepsilon < M(x,y) < \varepsilon + \delta \Rightarrow p(Ax, By) < \varepsilon$ (2)

Where$M(x,y) = max\left\{p(Sx,Ty), p(Ax,Sx), p(By,Ty), \frac{1}{2}(p(Sx,By) + p(Ax,Ty))\right\}$

(C_3) for all $x, y \in X$ with $M > 0 \Longrightarrow p(Ax, By) < M(x,y)$

(iii) $p(Ax, By) < max\{a[p(Sx,Ty) + p(Ax,Sx) + p(By,Ty)], b[p(Sx,By) + p(Ax,Ty)]\}$

for$0 \le a < \frac{1}{2}, 0 \le b < \frac{1}{2}$
(3)

If one of $AX, BX, SX and TX$is a closed subset of X, then

(i) $A$and$S$ have a coincidence

(ii)$B and T$have coincidence

Moreover, if A and S, as well as, B and T are weakly compatible, then A, B, S and T have a unique common fixed point.

In **2013**Abdeljawad T.[4] developed the fixed point theorems for α-contractive type maps to Meir-Keeler versions and generalize the resultsas:

Let (X, d)be an (f, g)orbitally complete metric space, where f, g are self-mappings of X. Also, letα: $X \times X \to [0, \infty)$ be a mapping. Assume the following:

(1) (f, g) is α-admissible and there exists an $x_0 \in X$such that

$\alpha(x_0, fx_0) \ge 1\alpha(x_0, fx_0) \ge 1$.

(2) the pair (f, g) is generalized Meir-Keeler α-contractive.

Then the sequence $d_n = d(x_n, x_{n+1})$ is monotone decreasing. If, moreover, we assume that

(3) On the (f, g)-orbit ofx_0 we have $\alpha(x_n, x_j) \ge 1$ for all n even and $j > n$ odd and that $f$and g are continuous on the (f, g)-orbit ofx_0.

Then either (1) f or g has a fixed point in the (f,g)–orbit$\{x_0\}$ofx_0 or (2) f and g have a common fixed point $p$and $\lim_{n\to\infty} x_n = p$. If, moreover, we assume that the following condition (H) holds: If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ implies $\alpha(x_n, x) \geq 1$for all n, then uniqueness of the fixed point is obtained.

Definition 1.1.[32] Let X be a non- empty set, and let $G: X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following axioms:

(G_1) $G(x,y,z) = 0 \; if \; x = y = z,$

(G_2) $0 < G(x,x,y)$, for all $x, y \in X, with \; x \neq y,$

(G_3) $G(x,x,y) \leq G(x,y,z), for\; all\; x, y, z \in X, with\; z \neq y,$

(G_4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (symmetry in all three variables),

(G_5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z), for\; all\; x,y,z,a \in X$,(Rectangle inequality).

Then the function G is called a generalized metric or more specifically a G-metric on X, and the pair (X,G) is called a G-metric space.

Example 1.2. Let R be the set of all real numbers. Define $G: R \times R \times R \to R^+$ by

$$G(x,y,z) = |x - y| + |y - z| + |z - x|, for\; all\; x,y,z \in X.$$

Then it is clear that (R,G) is a G-metric space.

Proposition 1.3.[32]Let (X,G) be a G-metric space. Then for anyx, y, z, and$a \in X$, it follows that

(1) $If\; G(x,y,z) = 0, then\; x = y = z,$
(2) $G(x,y,z) \leq G(x,x,y) + G(x,x,z),$
(3) $G(x,y,y) \leq 2G(y,x,x),$
(4) $G(x,y,z) \leq G(x,a,z) + G(a,y,z),$
(5) $G(x,y,z) \leq (\frac{2}{3})(G(x,y,a) + G(x,a,z) + G(a,y,z)),$
(6) $G(x,y,z) \leq (\frac{2}{3})(G(x,a,a) + G(y,a,a) + G(z,a,a)).$

Definition 1.4.[32]Let (X,G) be a G-metric space, let $\{x_n\}$) be a sequence of points of X, we say that $\{x_n\}$is G-convergent

$\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$;i.e.

for any $\epsilon > 0$, there exists $n_0 \in$ N such that

$G(x, x_n, x_m) < \epsilon$, for all $n, m \geq n_0$.

We refer to x as the limit of the sequence$\{x_n\}$ and write x_n (G) $\to x$.

Proposition 1.5. [32] Let (X,G) be a G −metric space. Then the following are equivalent:
(1) $\{x_n\}$is G-convergent to x.
(2) $G(x_n, x_n, x) \to 0 \; as\; n \to +\infty$
(3) $G(x_n, x, x) \to 0 \; as\; n \to +\infty$
(4) $G(x_n, x_m, x) \to 0 \; as\; m,n \to +\infty$

Definition 1.6. [32]Let (X,G) be a G-metric space. A sequence$\{x_n\}$ is called G-Cauchy if given $\epsilon > 0$, there is $n_0 \in$ N such that $G(x, x_n, x_m) < \epsilon$, for all n, m, l $\geq n_0$ that is

if$G(x_n, x_m, x_l) \to 0$ as n, m, l $\to \infty$.

Proposition 1.7. [32] In a G −metric space(X,G), the followings are equivalent-

(1) The sequence $\{x_n\}$ is G − Cauchy.
(2) For every $\epsilon > 0$, $\exists N \in$ N such that $G(x_n, x_m, x_m) < \varepsilon$ for all $n, m \geq N$.

(3$\{x_n\}$is a Cauchy sequence in the metric space (X, G).

Definition 1.8. [32] A G −metric space (X, G) is said to be G-complete (or a complete G-metric space) if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Example 1.9. Let $X = \{0, 1, 2, 3. \dots\}$ and $G : X \times X \times X \to R^+$ be defined as follows:

$$G(x,y,z) = \begin{cases} x + y + z, if\ x, y, z\ are\ all\ distinct\ and\ different\ from\ zero \\ x + z, if\ x = y \neq z\ and\ all\ are\ different\ from\ zero \\ y + z + 1, if\ x = 0, y \neq z, y\ and\ z\ are\ different\ from\ zero \\ y + 2, if\ x = 0, y = z \neq 0 \\ z + 1, if\ x = 0, y = 0, z \neq 0 \\ 0, if\ x = y = z \end{cases}$$

Then (X, G) is a complete G-metric space,then$\boldsymbol{G}$ is also nonsymmetric since $G(0,0,1) \neq G(1,1,0)$.

Proposition 1.10.[32] In a G −metric space, (X, G), the following are equivalent.

(1) thesequence$\{x_n\}$) is G-Cauchy.

(2) For every ϵ> 0, there exists N ∈ N such that $G(x_n, x_m, x_m)$<ϵ, for all $n, m \geq N$.

(3) $\{x_n\}$is a Cauchy sequence in the metric space (X, d_G).

Definition 1.11. A G −metric space (X, G) is said to be G-complete (or a complete G-metric space) if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Corollary 1.12. Every G-convergent sequence in a G −metric space is G-Cauchy.

Corollary 1.13. If a G-Cauchy sequence in a G-metric space (X, G) contains a G −convergent subsequence, then the sequence itself is G-convergent.

Proposition 1.14. A G-metric space (X, G) is G-complete if and only if(X, d_G) is a complete metric space.

Corollary 1.15. IfY is a non-empty subset of a G −complete metric space (X, G), then $(Y, G|Y)$ is complete if and only if Y is G −closed in (X, G).

Definition 1.16. Let $(X, G)$and (X', G') be G-metric spaces and let $f : (X, G) \to (X', G')$ be a function, then f is said to be G-continuous at a point $a \in X$ if given $\varepsilon > 0$, there exists δ > 0 such that $x, y \in X$; $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \varepsilon$.

A function f is G −continuous on X if and only if it is G-continuous at all $a \in X$.

Proposition 1.17. Let (X, G) and(X', G') be G-metric spaces and let$f : (X, G) \to (X', G')$ is G −continuous at a point $x \in X$ if and only if G −sequentially continuousat $x \in X$; that is whenever $\{x_n\}$ is G −convergent to x we have $(f(x_n))$ is G −convergent to $(f(x))$.

Definition1.18. Let (X, d) be a metric space and T be a self map on X.Then T is called a Meir-Keeler type contraction whenever for each $\varepsilon > 0$, there exists $\delta > 0$ *such that*

$$\varepsilon < d(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) < \varepsilon$$

This contraction has been modified and extended by many authors in metric spaces and some other related structures[1-38, 40-41,43].Now we prove a unique fixed point result by using generalized Meir-Keeler contraction on G-metric spaces.

2.Main Results

Definition2.1.$Let\ (X,G) be\ a\ G$ −Metric spaceand T be a self map on X .Then T is called a generalized Meir-Keeler type contraction whenever for each $\varepsilon > 0$, there exists $\delta > 0$ *such that*

$\varepsilon < M(x,y,z) < \varepsilon + \delta \Rightarrow G(Tx,Ty,Tz) < \varepsilon$
(2.1)

Where$M(x,y,z) = max\{G(x,y,z), G(Tx,x,x), G(x,Ty,z), G(x,y,Tz)\}$

Remark 2.2. Note that if T is a generalized Meir-Keeler type contraction then we have

$G(Tx,Ty,Tz) < M(x,y,z).$
(2.2)

Now we come to our main results.

Proposition2.3.. $Let\ (X,G)\ be\ a\ G$ −Metric space and let $T: X \to X$ be a generalized Meir-Keeler type contraction. Then

$\lim_{n\to\infty} G(T^{n+1}x, T^n x, T^n x) = 0$for all $x \in X$.

Proof: Let $x_0 \in X$.we define an iterative sequence $\{x_n\}$ as

$x_n = Tx_n = T^{n+1}x_0$
(2.3)

for all $n \geq 0$. If $x_{n_0+1} = x_{n_0}$ for some $n_0 \geq 0$, then x_{n_0} is required fixed point of T. Indeed ,

$Tx_{n_0} = x_{n_0+1} = x_{n_0}$. In this case, the proposition follows. Throughout the proof, we assume that $x_{k+1} \neq x_k$for all $k \geq n_0$.consequently, we have $M(x_{n+1}, x_n, x_n) > 0$ for every$n \geq 0$. By remark 2.2, we get

$$G(x_{n+2}, x_{n+1}, x_{n+1}) = G(Tx_{n+1}, Tx_n, Tx_n) < M(x_{n+1}, x_n, x_n)$$
$$= max\{G(x_{n+1}, x_n, x_n), G(Tx_{n+1}, x_{n+1}, x_{n+1}), G(x_{n+1}, Tx_n, x_n), G(x_{n+1}, x_n, Tx_n)\}$$
$$= max\{G(x_{n+1}, x_n, x_n), G(x_{n+2}, x_{n+1}, x_{n+1})\}$$

Since $M(x_{n+1}, x_n, x_n) > 0$ for each n, we find that

$$G(x_{n+2}, x_{n+1}, x_{n+1}) < M(x_{n+1}, x_n, x_n) \leq max\{G(x_{n+1}, x_n, x_n), G(x_{n+2}, x_{n+1}, x_{n+1})\}$$

By the use of remark 2.2 again,we notice that the case where

$$max\{G(x_{n+1}, x_n, x_n), G(x_{n+2}, x_{n+1}, x_{n+1})\} = G(x_{n+2}, x_{n+1}, x_{n+1})$$

is impossible . Hence we derive that

$$G(x_{n+2}, x_{n+1}, x_{n+1}) < M(x_{n+1}, x_n, x_n) = G(x_{n+1}, x_n, x_n) \quad (2.4)$$

for every n.

Thus $\{G(x_{n+1}, x_n, x_n)\}_{n=0}^{\infty}$ is decreasing sequence which is bounded by 0. Hence, it converges to some $\varepsilon \in [0, \infty)$,i.e.

$$\lim_{n\to\infty} G(x_{n+1}, x_n, x_n) = \varepsilon \quad (2.5)$$

In particular, we have

$$\lim_{n\to\infty} M(x_{n+1}, x_n, x_n) = \varepsilon \quad (2.6)$$

Here $\varepsilon = inf\{G(x_{n+1}, x_n, x_n): n \in N\}$

We claim that $\varepsilon = 0$. Suppose to the contrary that $\varepsilon > 0$. Regarding (2.6) together with the assumption that T is a generalized Meir- Keeler type contraction, for this , there exists a $\delta > 0$and a natural no m such that

$\varepsilon \leq M(x_{m+1}, x_m, x_m) < \varepsilon + \delta$implies

$$G(Tx_{m+1}, Tx_m, Tx_m) = G(x_{m+2}, x_m, x_m) < \varepsilon \quad (2.7)$$

which is a contradiction Because $\varepsilon = \inf\{G(x_{n+1}, x_n, x_n): n \in N\}$.

Theorem2.4. : Let (X,G) be a complete G −metric space and $T: X \to X$ be an orbitally continuous generalized Meir- Keeler type contraction. Then T has a unique fixed point, say $w \in X$. ver,$\lim_{n\to\infty} G(T^n x, w, w) = 0$ for all $x \in X$.

Proof: Let $x_0 \in X$.we define an iterative sequence $\{x_n\}$ as

$$x_n = Tx_n = T^{n+1}x_0 \tag{2.8}$$

for all $n \geq 0$. We claim that $\lim_{m,n\to\infty} G(x_n, x_m, x_m) = 0$. If this is not the case , then there exist a $\varepsilon > 0$ and a subsequence $\{x_{n(i)}\}$ of $\{x_n\}$ such that

$$G(x_{n(i)}, x_{n(i+1)}, x_{n(i+1)}) > 2\varepsilon \tag{2.9}$$

For the same $\varepsilon > 0$above,$\exists\delta > 0$ such that

$$\varepsilon \leq M(x, y, z) < \varepsilon + \delta$$

which implies that $G(Tx, Ty, Tz) < \varepsilon$.

Set $r = \min\{\varepsilon, \delta\}$ and $g_n = G(x_n, x_n, x_{n+1})$ for all $n \geq 1$. By proposition 2.3, one can choose a natural no n_0 such that

$$g_n = G(x_n, x_n, x_{n+1}) < \frac{r}{4} \tag{2.10}$$

for all $n \geq n_0$. Let $n(i) > n_0$. We have $n(i) \leq n(i+1) - 1$.

If$G(x_{n(i)}, x_{n(i+1)-1}, x_{n(i+1)-1}) \leq \varepsilon + \frac{r}{2}$.

Then by using (G5)we derive

$$G(x_{n(i)}, x_{n(i+1)}, x_{n(i+1)}) \leq G(x_{n(i)}, x_{n(i+1)-1}, x_{n(i+1)-1}) + G(x_{n(i+1)-1}, x_{n(i+1)}, x_{n(i+1)})$$

$$\leq \varepsilon + \frac{r}{2} + g_{n(i+1)-1} < \varepsilon + \frac{3r}{4} < 2\varepsilon \tag{2.11}$$

which contradicts the assumption(2.9). Therefore, there are values of k such that

$n(i) \leq k \leq n(i+1)$and $G(x_{n(i),} x_k, x_k) > \varepsilon + \frac{r}{2}$.

Now if$(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}) \geq \varepsilon + \frac{r}{2} > r + \frac{r}{2} > \frac{r}{4}$.

This is a contradiction because of (2.10). Hence there are values of k with $n(i) \leq k \leq n(i+1)$ such that

$G(x_{n(i),} x_k, x_k) < \varepsilon + \frac{r}{2}$.

We choose the smallest integer k with $k \geq n(i)$ such that $G(x_{n(i),} x_k, x_k) \geq \varepsilon + \frac{r}{2}$. Then, we find

$G(x_{n(i),} x_{k-1}, x_{k-1}) < \varepsilon + \frac{r}{2}$.

So we see that

$$G(x_{n(i),} x_k, x_k) \leq G(x_{n(i),} x_{k-1}, x_{k-1}) + G(x_{k-1,} x_{k-1}, x_k) < \varepsilon + \frac{r}{2} + \frac{r}{4} = \varepsilon + \frac{3r}{4} \tag{2.12}$$

Now we can choose a natural number k satisfying $n(i) \leq k \leq n(i+1)$ such that

$$\varepsilon + \frac{r}{2} \leq G(x_{n(i),} x_k, x_k) < \varepsilon + \frac{3r}{4} \tag{2.13}$$

Therefore we obtain the inequalities

$$G(x_{n(i),} x_k, x_k) < \varepsilon + \frac{3r}{4} < \varepsilon + r, \tag{2.14}$$

$$G(x_{n(i),} x_{n(i)+1}, x_{n(i)+1}) = g_{n(i)} < \frac{r}{4} < \varepsilon + r, \tag{2.15}$$

$$\text{and} G(x_k, x_{k+1}, x_{k+1}) = d_k < \frac{r}{4} < \varepsilon + r \tag{2.16}$$

By (2.14) -(2.16),we get that$M(x_{n(i),} x_k, x_k) < \varepsilon + r \leq \varepsilon + \delta$.Since$T$ is a generalized Meir-Keeler type contraction,

$$G(x_{n(i)+1,} x_{k+1}, x_{k+1}) < \varepsilon$$

By using (G5), we obtain

$$G(T^{n(i)}x_0, T^k x_0, T^k x_0) \leq G(T^{n(i)}x_0, T^{n(i)+1}x_0, T^{n(i)+1}x_0) + G(T^{n(i)+1}x_0, T^k x_0, T^k x_0)$$

$$\leq G(T^{n(i)}x_0, T^{n(i)+1}x_0, T^{n(i)+1}x_0) + G(T^{n(i)+1}x_0, T^{k+1}x_0, T^{k+1}x_0) + G(T^{k+1}x_0, T^k x_0, T^k x_0)$$

We combine the inequality above with (2.13), (2.15) and (2.16) to conclude

$G(x_{n(i)+1,}x_{k+1}, x_{k+1}) \geq G(x_{n(i),}x_k, x_k) - G(x_{k,}x_{k+1}, x_{k+1}) > \varepsilon + \frac{r}{2} - \frac{r}{4} = \varepsilon.$

which is a contradiction. Therefore our claim is proved. So $\{x_n\} = \{T^n x_0\}$ is a $G-$ Cauchy sequence. Since (X, G) is $G-$complete, the sequence $\{x_n\}$ converges to some $w \in X$.

By proposition 2.3, we have

$$\lim_{n\to\infty} G(T^n x_0, w, w) = \lim_{n\to\infty} G(T^n x_0, T^n x_0, w) = 0.$$

Next we will prove w is a fixed point of T.

Since T is orbitally continuous and $\lim_{n\to\infty} G(T^n x_0, w, w) = 0$, we get

$$\lim_{n\to\infty} G(TT^n x_0, Tw, Tw) = \lim_{n\to\infty} G(x_{n+1}, Tw, Tw) = 0$$

where$x_{n+1} = TT^n x_0 = T^{n+1}x_0$. Thus $\{x_{n+1}\}$ converges to Tw in (X, G). By the uniqueness of limit , we get $Tw = w$.

Finally we show that T has a unique fixed point. If there exist $u \in X$ such that $Tu = u$ and $G(u, w, w) \neq 0$, then we get $M(u, w, w) \geq G(u, w, w) > 0$.Since$T$ is a generalized Meir-Keeler type contraction, we derive

$$\begin{aligned} 0 < G(u, w, w) = G(Tu, Tw, Tw) &< M(u, w, w) \\ &= max\{G(u, w, w), G(Tu, u, u), G(u, Tw, w), G(u, w, Tw)\} = \max\{G(u, w, w), 0\} \\ &= G(u, w, w) \end{aligned}$$

which is a contradiction. Thus we find that $G(u, w, w) = 0$.So by (G1) we conclude that $u = w$. In lar,T has a unique fixed point.

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