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Comparative Analysis of Finite Difference Methods for Solving Second Order Linear Partial Differential Equations

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Abstract
The Liebmann's and Gauss Seidel finite difference methods of solution are applied to a two dimensional second order linear elliptic partial differential equation with specified boundary conditions.The analytical (exact) solution obtained shows that the error of the numerical solutions increases with the number of iterations and consequentlyhas strong effects on the accuracy of the solution.
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1. INTRODUCTION

The analytical or classical solution of partial differential equations is the ability to write down a general formula of the problem as a solution and if possible to show that, the solution exists and depends continuously on the data given in the problem [2].This will imply that the solution obtained must be real, analytic or at least infinitely differentiable, in other to satisfy the given problem

Certain types of boundary value problems can be solved by replacing the differential equations by the corresponding difference equation and then solving the latter by a process of iteration. This method was devised and first used by L. T. Richardson and it was later improved by H. Liebmann.Finite difference methods are numerical methods for approximating the solutions to differential equations using finite difference equations to approximate derivatives ([1], [3]).

An important aspect of numerical analysis of partial differential equations is the numerical solution of the finite linear algebraic systems that are generated by the discrete equations. These are in general very large, but with sparse matrices, which makes iterative methods suitable. The development of convergence analysis for such methods has paralleled that of the error analysis sketched above. In the 1950s and 1960s particular attention was paid to systems associated with finite difference approximation of positive type of second order elliptic equations, particularly the five-point scheme, and starting with the Jacobi and Gauss-Seidel methods techniques were developed such as the Frankel and Young successive over-relaxation and the Peaceman-Rachford alternating direction methods ([4}, [5], [6], [7]).

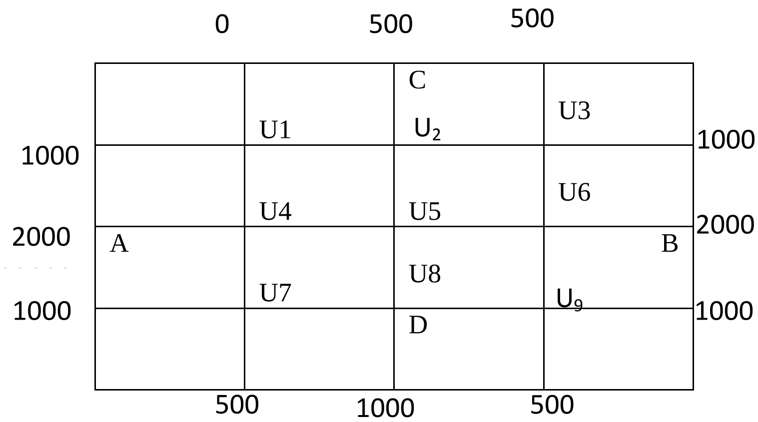
2. STATEMENT OF THE PROBLEM

In this work, we investigate the numerical solution of the Laplacian equation $\nabla^2 u = 0$ using two methods namely, Leibmann's iteration process and gauss Seidel iteration process with a view to comparing the results obtained with the exact solution.

3. MAIN RESULTS

3.1 LIEBMANN'S ITERATION PROCESS

Given the mesh below with boundary values as shown, we solve the equation $\nabla^2 u = 0$in two dimensions using Leibmann's iteration process.



SOLUTION:
Take the central horizontal and vertical lines as AB and CD respectively. Let $u_1, u_2, \dots, u_9$ be the values of u at the interior grid points of the mesh. The values of u on the boundary are symmetrical w. r. t. the lines AB about AB and CD.

$\therefore u_1 = u_3 = u_7 = u_9; \quad u_2 = u_8; \quad u_4 = u_6$ and u_5 is not equal to any value.

$\therefore$ it is enough if we find u_1, u_2, u_4, and u_5.

Rough values of u's

$U_5 = \frac{1}{4}(1000 + 2000 + 1000 + 2000) = 1500$ (SFPF)

$U_1 = \frac{1}{4}(0 + 1500 + 2000 + 1000) = 1125$ (DFPF)

$U_2 = \frac{1}{4}(1000 + 1500 + 1125 + 1125) = 1187.5 \cong 1188$ (SFPF)

$U_3 = \frac{1}{4}(0 + 1500 + 1000 + 2000) = 1125$ (DFPF)

$U_4 = \frac{1}{4}(1125 + 1125 + 2000 + 1500) = 1437.5 \cong 1438$ (SFPF)

$U_6 = \frac{1}{4}(1125 + 1125 + 1500 + 2000) = 1437.5 \cong 1438$ (SFPF)

$u_7 = \frac{1}{4}(0 + 1500 + 1000 + 2000) = 1125$ (DFPF)

$U_8 = \frac{1}{4}(1000 + 1125 + 1125 + 1500) = 1187.5 \cong 1188$ (SFPF)

SO, $u_1 = u_3 = u_9 = u_7 = 1125$

$$u_2 = u_8 = 1188$$
$$u_4 = u_6 = 1438$$

Now, we have got the rough values at all interior grid points and already we possess the boundary values at the lattice points. We will now improve the values by using the standard five point formula.

FIRST ITERATION:

$$u_1^{(n+1)} = \frac{1}{4}(1000 + 500 + u_2 + u_4) = \frac{1}{4}(1000 + 500 + 1188 + 1438) = 1032$$
$$u_1^{(1)} = u_3^{(1)} = u_9^{(1)} = u_7^{(1)}$$
$$u_2^{(n+1)} = \frac{1}{4}(u_1 + u_3 + u_5 + 1000) = \frac{1}{4}(1032 + 1032 + 1500 + 1000) = 1141$$
$$u_2^{(n+1)} = u_8^1$$
$$u_4^{(n+1)} = \frac{1}{4}(u_1 + u_5 + 2000 + u_7) = \frac{1}{4}(1032 + 1500 + 2000 + 1032) = 1391$$
$$u_4^{(n+1)} = u_6$$
$$u_5^{(n+1)} = \frac{1}{4}(u_2 + u_8 + u_4 + u_6) = \frac{1}{4}(1141 + 1141 + 1391 + 1391) = 1266$$

SECOND ITERATION

$$U_1^{(2)} = \frac{1}{4}(1000 + 500 + 1141 + 1391) = 1008$$
$$U_1 = U_3 = U_9 = U_7$$
$$U_2^{(2)} = \frac{1}{4}(1008 + 1008 + 1266 + 1000) = 1069$$
$$U_2^{(2)} = U_8^{(2)}$$
$$U_4^{(2)} = \frac{1}{4}(1008 + 1266 + 2000 + 1008) = 1321$$
$$U_4^{(2)} = U_6^{(2)}$$
$$U_5^{(2)} = \frac{1}{4}(1069 + 1069 + 1321 + 1321) = 1195$$

SIMILARLY,

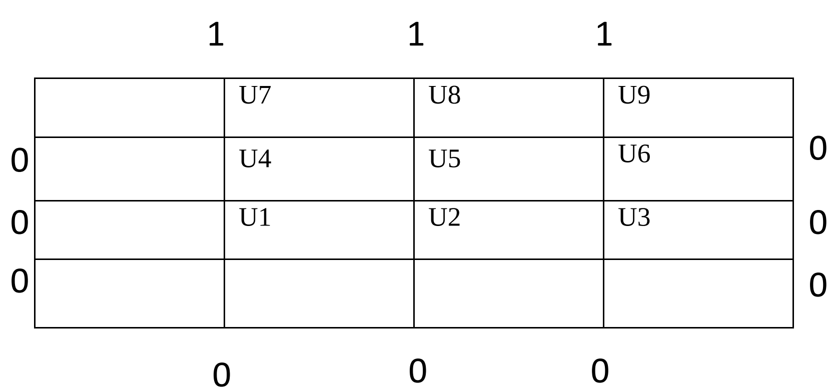
ITERATION	$u_1 = u_3 = u_9 = u_7$	$u_2 = u_8$	$u_4 = u_6$	u_5
Third	973	1035	1288	1162
Fourth	956	1019	1269	1144
Fifth	947	1010	1260	1135
Sixth	942	1005	1255	1130
Seventh	940	1003	1253	1128
Eighth	939	1002	1252	1127
Ninth	939	1001	1251	1126

Very small difference is in the eighth and ninth iteration. Thus,

$$u_1 = u_3 = u_7 = u_9 = 939$$
$$u_2 = u_8 = 1001$$
$$u_4 = u_6 = 1251$$
$$u_5 = 1126$$

3.2 GAUSS SEIDEL ITERATION

We now solve the same problem $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in the domain of the figure given below by Gauss' seidel iteration method.



SOLUTION

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

The above equation is a second order partial differential equation. It is an elliptic PDE. It has two dimensions.
To solve the above equation using finite difference method, the second order PDE must first be discretize. Using the central difference approximation.

Recall, $\frac{\partial^2 u}{\partial x^2}=\frac{u_{i+1}-2u_{i,j}+u_{i-1,j}}{hx^2}$

$$\frac{\partial^2 u}{\partial y^2}=\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{hy^2}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{hx^2}+\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{hy^2}$$

Let $hx=hy=h$

$$=\frac{u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}-4u_{i,j}}{h^2}$$

$$u_{i,j}=u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}$$

$$u_{i,j}=\frac{u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}}{4}$$

Then initial approximations are; $u_1=u_2=u_3=u_4=u_5=u_6=u_7=u_8=u_9=0$
Then we perform 10 successive iterations which are given below;
FIRST ITERATION

$$U_1=\frac{1}{4}(0+U_4+U_2+0)U_1=\frac{1}{4}(0+0+0+0)=0$$

$$U_2=\frac{1}{4}(U_1+U_5+U_3+0)U_2=\frac{1}{4}(0+0+0+0)=0$$

$$U_3=\frac{1}{4}(U_2+U_6+0+0)U_3=\frac{1}{4}(0+0+0+0)=0$$

$$U_4=\frac{1}{4}(0+U_7+U_5+U_1)U_4=\frac{1}{4}(0+0+0+0)=0$$

$$U_5=\frac{1}{4}(U_4+U_8+U_6+U_2)U_5=\frac{1}{4}(0+0+0+0)=0$$

$$U_6=\frac{1}{4}(\mathrm{U}_5+U_9+0+U_3)U_6=\frac{1}{4}(0+0+0+0)=0$$

$$U_7=\frac{1}{4}(0+1+U_4+U_8)U_7=\frac{1}{4}(0+0+1+0)=0$$

$$U_8=\frac{1}{4}(U_7+1+U_9+U_5)U_8=\frac{1}{4}(0.25+1+0+0=0.312)$$

$$U_9=\frac{1}{4}(U_8+1+0+U_6)\qquad U_9=\frac{1}{4}(0.312+1+0+0)=0.328$$

SECOND ITERATION

$$U_1^{(2)}=\frac{1}{4}(0+0+0+0)=0$$

$$U_2^{(2)}=\frac{1}{4}(0+0+0+0)=0$$

$$U_3^{(2)}=\frac{1}{4}(0+0+0+0)=0$$

$$U_4^{(2)}=\frac{1}{4}(0+0.25+0+0)=0.062$$

$$\boldsymbol{U_5^{(2)}}=\frac{1}{4}(\mathbf{0}+0.312+0+0)=0.078$$

$$U_6^{(2)}=\frac{1}{4}(U_5+U_9+0+U_3=0.082)$$

$$U_7^{(2)}=\frac{1}{4}(0+1+0.0312+0)=0.328$$

$$U_8^{(2)} = \frac{1}{4}(0.25 + 1 + 0 + 0.328) = 0.394$$
$$U_9^{(2)} = \frac{1}{4}(0 + 0.312 + 1 + 0) = 0.328$$

ITERATION	U1	U2	U3	U4	U5	U6	U7	U8	U9
I	0	0	0	0	0	0	0.25	0.312	0.328
2	0	0	0	0.062	0.078	0.082	0.328	0.394	0.328
3	0.016	0.024	0.027	0.106	0.100	0.127	0.375	0.398	0.464
4	0.32	0.053	0.045	0.0140	0.196	0.160	0.401	0.499	0.415
5	0.048	0.072	0.058	0.161	0.223	0.174	0.415	0.513	0.0422
6	0.058	0.085	0.065	0.174	0.236	0.181	0.058	0.520	0.425
7	0.065	0.092	0.068	0.181	0.244	0.184	0.425	0.525	0.427
8	0.068	0.095	0.070	0.184	0.247	0.186	0.427	0.525	0.428
9	0.070	0.097	0.071	0.186	0.249	0.187	0.428	0.526	0.428
10	0.071	0.098	0.071	0.187	0.250	0.187	0.428	0.526	0.428

ANALYTIC SOLUTION

We consider the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \qquad (1)$$

Let $u = X(x).Y(y)$ (2)

Substituting the values of $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ in equation (1), we have,

$$x''y + xy'' = 0$$

or $\frac{x''}{x} = -\frac{y''}{y} = -p^2$ for some real p.

From here, we have that

$$X'' = -p^2X \quad \text{or} \quad X'' + p^2X = 0 \tag{3}$$

$$Y'' = p^2Y \quad \text{or} \quad Y'' - p^2Y = 0 \tag{4}$$

Auxiliary equation corresponding to equation (3) is

$$m^2 + p^2 = 0$$

$$=> m = \pm\sqrt{-p^2} = \pm ip$$

$$X = C_1 \cos px + C_2 \sin px.$$

Auxiliary equation corresponding to equation (4) is

$$m^2 + p^2 = 0$$

$$=> m = \pm\sqrt{-p^2} = \pm p$$

$$Y = C_3e^{py} + C_4e^{-py}.$$

Substituting the values of X and Y in equation (2) we have,

$$U = (C_1 \cos px + C_2 \sin px)(C_3e^{py} + C_4e^{-py}). \tag{5}$$

Substituting$x = 0$ and $U = 0$ in equation (5) we have,

$$0 = C_1(C_3e^{py} + C_4e^{-py})$$

Equation (5) reduces to

$$U = (C_2 \sin px)(C_3e^{py} + C_4e^{-py})$$

$$0 = C_2 \sin 4p\,(C_3e^{py} + C_4e^{-py})$$

(6)

$$C_2 \neq 0$$

$$\therefore \sin 4p = 0 = \sin n\pi$$

This implies that

$$4p = n\pi \quad \text{or} \quad p = \frac{n\pi}{4}$$

Now equation (6) becomes

$$U = C_2 \sin\frac{n\pi x}{4}\left(C_3e^{\frac{n\pi}{4}y} + C_4e^{-\frac{n\pi}{4}y}\right) \tag{7}$$

Substituting$y = 0$ and $U = 0$ in equation (7) we have,

$$0 = C_2 \sin\frac{n\pi x}{4}(C_3 + C_4).$$

This means that

$$C_3 + C_4 = 0 \quad \text{or} \quad C_3 = -C_4.$$

Equation (7) becomes

$$U = C_2C_3 \sin\frac{n\pi x}{4}\left(e^{\frac{n\pi}{4}y} - e^{-\frac{n\pi}{4}y}\right) \quad . \tag{8}$$

Substituting$y = 4$and $U = 1$in equation (8) we get,

$$1 = \sin\frac{\pi}{2} = C_2C_3 \sin\frac{n\pi x}{4}\left(e^{\frac{n\pi}{4}4} + e^{-\frac{n\pi}{4}4}\right), \tag{9}$$

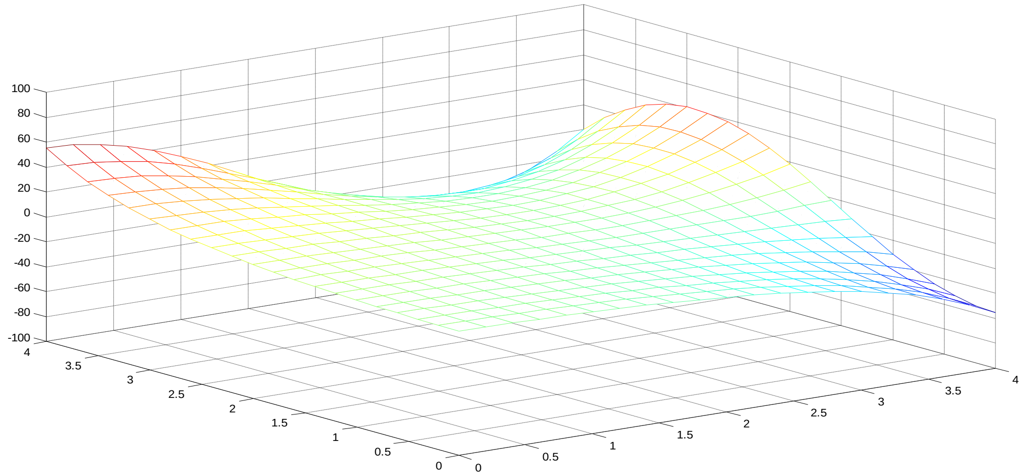
i.e., $C_2C_3 = \frac{1}{\sin\frac{n\pi x}{4}(e^{n\pi} + e^{-n\pi})}$

Substituting these values in equation (8) we have,

$$u = \frac{1}{\sin\frac{n\pi x}{4}(e^{n\pi} + e^{-n\pi})} \sin\frac{n\pi x}{4}\left(e^{\frac{n\pi}{4}y} - e^{-\frac{n\pi}{4}y}\right),$$

$$u = \frac{\left(e^{\frac{n\pi}{4}y} - e^{-\frac{n\pi}{4}y}\right)}{(e^{n\pi} + e^{-n\pi})},$$

$$u = \frac{\sinh\frac{n\pi}{4}y}{\sin\, n\pi}.$$



The figure shows the results of the solution of an elliptic partial differential equation in consideration.

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