Optimal Control of Customers to the Service Facility with Two Types of Customers

C.Selvakumar[#1], P.Maheswari[*2], C.Elango[#3]
[#1,2,] *Department of Mathematical Sciences,*
Cardamom Planters' Association College, Bodinayakanur, Tamilnadu, India.
[#3,]*Associate Professor,Department of Mathematical Sciences,*
Cardamom Planters' Association College, Bodinayakanur, Tamilnadu, India.

Abstract

In this article, we considered a discrete-time service facility system, viewed as a Markov Decision Process(MDP). Decisions are taken at discrete time epochs to control admissions to the system. Here the queue before the server is divided into eligible queue and potential queue. Potential queue has two types of customers (Priority and non-priority). It is assumed that demands arrived throughout the period but they are satisfied only at the end of the period. The MDP based on average cost criteria is used to find the optimal policy to be implemented for the system. Numerical example is provided to illustrate the problem.

Keywords

Markov Decision Processes, Admission Control, Discrete time Service Facility System, Two types of customers, Average Cost Criteria.

I. Introduction

Modelling of inventory systems maintained at a service facility has received considerable attention in the last three decades Berman et al. [1] studied the first model in inventory management with a service facility which is releasing one item from inventory to complete each service. This is equivalent to the make-to-order production system with common component inventory. They considered a model with constant demand and service rate in which queues can occur only during stock out period. So treating the queue in the service facility as a potential component whose length shall be reduced. So optimal admission control must be done to protect the system from congestion.

For this purpose we imposed the Markov Decision Process frame on this problem to implement sequential decision making. This kind of decision problems arise in feed back control of engineering systems, portfolio management and supply chain management etc.

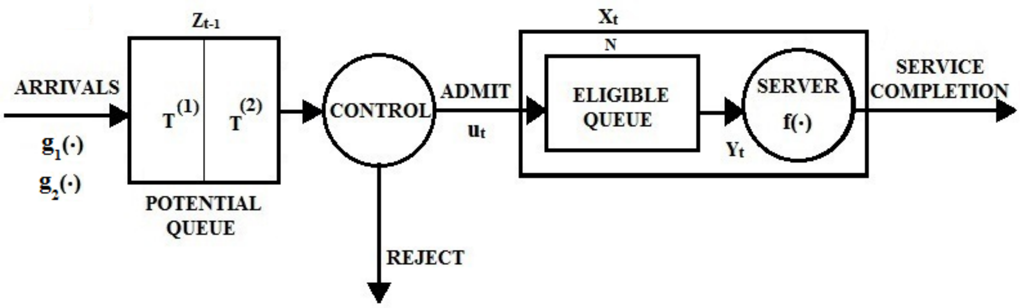
The standard mathematical formulation of this problem involves MDPs. Thus the states of the system is modeled as a Markov Chain, whose transition probabilities depends on the appropriate action chosen by considering the state-action dependent cost is incurred at each stage.

Recently Kim [6] considered the admission control and the inventory management problem of a make-to-order (MTO) facility with a common component, which is purchased from a supplier under stochastic lead time processes and setup costs. Arriving demands of MTO type (different types) are satisfied by using common (single) component. Selvakumar et.al [10] considered a discrete time MDP in a service facility system in which inventory is maintained to complete the service. Decision are taken at discrete time epochs to control both admission and inventory control in service facility systems. Control system is used to transfer customers from potential queue to eligible queue, but with single demand class.

When arriving customers consist of two types (ordinary and priority) as already studied (Veinott [12], Nahmias and Demmy [7], Ha ([3], [4]), Dekker, Hill, and Kleijn [2], Sapna [9], Karthick et al. [5]).

In this article, we considered a service facility system with two types of customers. The arrival of customers to the system is controlled by taking decisions at discrete decision epochs. Here, we use policy iteration method to optimize the expected total cost rate. In the last section a numerical example is provided to illustrate the model.

II. MODEL DESCRIPTION



- The system is observed every $\eta > 0$ unit of time and the decision epochs are $0, \eta, 2\eta, \ldots$
- Admissions to the service facility is controlled, by splitting the queue into Eligible queue and Potential queue. Potential queue has two types of customer called priority (**$T^{(1)}$**) and non-priority (**$T^{(2)}$**) customers.
- At each decision epoch the controller observes the number of customers in the system (Eligible queue + Server) and number of priority and non-priority customers in the potential queue.
- Assume that the maximum capacity of waiting space in the eligible queue is N (finite).
- Maximum number of customers to be admitted at time epoch $t = N$ - Number of customer in the eligible queue at time t. Other customers are rejected.
- Arriving customers to service facility system follows a probability distribution $g_1(\cdot)$ and $g_2(\cdot)$ for priority and non-priority customers respectively and the arriving customers are placed in potential queue. Possible service completions in each period follows a probability distribution $f(\cdot)$.
- No partial service completion allowed during any period.
- All serviced customers depart the system at the end of period.

III. MODEL FORMULATION

A We consider the MDP having five components (tuples) $(T, S, A_s, p_t(\cdot|), r_t(\ ))$.

Decision Epochs:

$$T = \{0, \eta, 2\eta, \ldots\}.$$

States:

$$S = S_1 \times S_2$$

$$S = \{0, 1, 2, \ldots, N\} \times \{0, 1, 2, \ldots\}.$$

$$S_1 = \{s_1 : s_1 \text{ denotes the number of customers in the system } (\text{eligible queue} + \text{server})\}$$

$$S_2 = \{s_2 : s_2 \text{ denotes the number of customers in the potential queue}\}$$

Actions:

Number of customers admitted is the decision variable

$$A_{(s_1, s_2)} = \{0, 1, 2, \ldots, s_2\}, \quad s_1 + s_2 \le N.$$

Transition Probability:

$$p_t(s' | s, a) = \begin{cases} f(s_1 + a - s_1')\, g(s_2') & \text{if } a + s_1 > s_1' > 0 \\ \left[\sum_{i = s_1 + a}^{N} f(i) \right] g(s_2') & \text{if } s_1' = 0,\ a + s_1 > 0 \\ g(s_2') & \text{if } s_1' = a + s_1 = 0 \\ 0 & \text{if } s_1' > a + s_1 \ge 0. \end{cases}$$

where $g(s_2') = g_1(n_1')\cdot g_2(n_2'),\ s = (s_1, s_2),\quad s' = (s_1', s_2')$.

Cost:

$$c_t(s,a) = k(s_1 + a) + p(i),\quad a \in A = \bigcup_{s\in S} A_s,\ \ s = (s_1, s_2).$$

The stationary cost structure consist of two components: a waiting cost $k(y)$ per period when there are $y(= s_1 + a)$ customers in the system and an incentive cost $p(i)$, when i priority customers are transfer from potential queue to the system.

IV. ANALYSIS

Let X_t denote the number of customers in the system immediately prior to the decision epoch t and Z_t is the number of customers arrivals in the period t. Customer arriving in the period $t-1$ enter the potential queue at time epoch t. Potential queue has two types of customers (Priority and non-priority). $X_t^{(1)}$ and $X_t^{(2)}$ represents number of priority and non-priority customers in the potential queue at time epoch t respectively.

Decision Rule:

At the decision epoch t the controller admits $(N - X_t)^+$ (number of waiting space in the system at time epoch t) $=\ u_t$ of customers from the potential customer queue into the system.

$$(x)^+ = \begin{cases} x & if \quad x > 0 \\ 0 & if \quad x \le 0. \end{cases}$$

The random variable Z_t assumes non-negative values which follows a time invariant probability distribution $g(n)$,

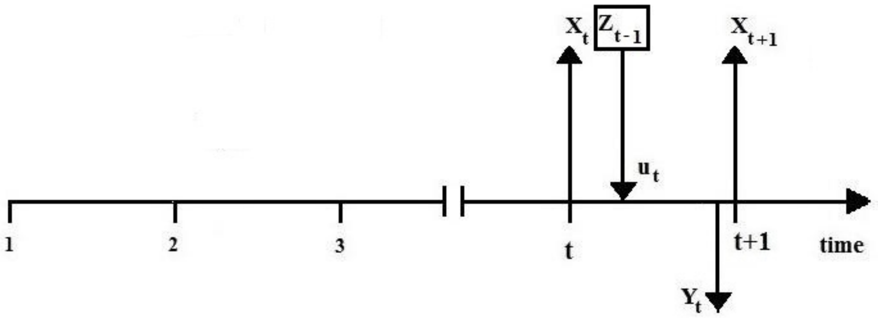
$$\begin{aligned} g(n) &= g_1(n_1)\cdot g_2(n_2) \\ &= \Pr\{Z_t^{(1)} = n_1\}\Pr\{Z_t^{(2)} = n_2\},\ \ t = 0,1,2,\ldots \end{aligned}$$

where $Z_t^{(1)}$, $Z_t^{(2)}$ denotes the number of priority and non-priority customers arrived in the period t and g_1, g_2 are independent.

Let Y_t denote the number of "possible service completions" during period t. The random variable Y_t assume non-negative integer values and follows a time invariant probability distribution $f(n) = \Pr\{Y_t = n\},\ t = 0,1,2,\ldots$.

Time	Potential queue	System
t	$Z_{t-1}^{(1)} + Z_{t-1}^{(2)} = X_t^{(1)} + X_t^{(2)}$	X_t
$t+$	0	$X_t + u_t$

Here $t+$ denotes the time point in time immediately after the control has been implemented but prior to any service completions.



Number of customers admitted to system from potential job queue at time epoch *t*:

(i) If $X_t^{(1)} > N - X_t$ admit $(N - X_t)$ priority customers, reject all other customers including non-priority customers.

(ii) If $X_t^{(1)} < N - X_t$ admit $X_t^{(1)}$ customers and if $X_t^{(2)} > N - X_t - X_t^{(1)}$ admit $(N - X_t - X_t^{(1)})$ non-priority customers, reject remaining non-priority customers else if $X_t^{(2)} < N - X_t - X_t^{(1)}$ admit $X_t^{(2)}$ customers.

Here $t+$ denotes a point in time immediately after the control has been implemented but prior to any service completions.

The system state at a decision epoch *t* is denoted by the pair (X_t, I_t), where I_t denotes the content of the potential queue at decision epoch *t*.

The two component of the system state is given by

$$X_{t+1} = \begin{cases} X_t - Y_t + (N - X_t) & if \quad X_t^{(1)} > (N - X_t) \ \ and \ \ X_t^{(2)} \geq 0 \\ X_t - Y_t + X_t^{(1)} + (N - X_t - X_t^{(1)}) & if \quad X_t^{(1)} < (N - X_t) \ \ and \ \ X_t^{(2)} > (N - X_t - X_t^{(1)}) \\ X_t - Y_t + X_t^{(1)} + X_t^{(2)} & if \quad X_t^{(1)} < (N - X_t) \ \ and \ \ X_t^{(2)} < (N - X_t - X_t^{(1)}). \end{cases}$$

$I_{t+1} = Z_t^{(1)} + Z_t^{(2)} = X_{t+1}^{(1)} + X_{t+1}^{(2)}.$

The one step costs are given by, $c_t(s, a)$, $s = (s_1, s_2)$.

Let (X_t, I_t) denote the state of the system of decision epoch t (beginning of t^{th} period). Assume the stationary policy R and hence the transition probability

$$p_t(s' | s, a) = Pr\{(X_{t+1}, \mathrm{I}_{t+1}) = s' | (X_t, \mathrm{I}_t) = s, a\}, \quad s' = (s_1', s_2'), \ s = (s_1, s_2).$$

regardless the past history of the system up to time epoch t.

Then $\{(X_t; I_t) : t \geq 0\}$ is a Markov chain with discrete state space $S = S_1 \times S_2$. The t - step transition probabilities of the Markov chain under policy R is given by

$$p_t(s' | s)(R) = \Pr\{(X_t, \mathrm{I}_t) = s' | (X_0, \mathrm{I}_0) = s, R_a\}, \quad s' = (s_1', s_2'), \ s = (s_1, s_2).$$

Define $V_t(s, R)$, $s = (s_1, s_2)$ denote the total expected cost over the first t decision epochs with initial state (s_1, s_2) and policy R is adopted.

Then

$$V_t(\mathrm{s}, \mathrm{R}) = \sum_{k=0}^{t-1} \sum_{s' \in S} p^{(k)}(s, s')(R) c_{s'}(R), \quad s' = (s_1', s_2'), \ s = (s_1, s_2)$$

where,

$$C_s(R) = \text{waiting cost of customer/period + incentive cost.}$$

$$= C_3 \times \overline{L} + C_4 \times \overline{P},$$

where $\overline{L}$ denotes the mean number of customers in the eligible queue + 1 in service counter and $\overline{P}$ denotes the number of priority customers are transfer from potential queue to the system

V. COST ANALYSIS

The average cost function $h_s(R)$ is given by $h_s(R) = \lim_{t \to \infty} \frac{1}{t} V_t(\mathrm{s}, \mathrm{R})$, $(s_1, s_2) \in S$. The elements of the above average cost function is due to the Theorem (Puterman [8] & Tijms [11]).

Theorem 5.1

For all $s' = (s_1', s_2')$, $s = (s_1, s_2) \in S$, $\lim_{t \to \infty} \frac{1}{t} \sum_{k=1}^{t} p_t^{(k)}(s' | s, R)$ always exists and for any $s' = (s_1', s_2') \in \mathrm{S}$.

$$\lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{t} p^{(k)}(s'|s)=\begin{cases}\dfrac{1}{\mu_s'} & if \quad state\ s'\ is\ recurrent\\ 0 \quad if & state\ s'\ is\ transient,\end{cases}$$

where μ_s' denote the mean recurrent time from state (s_1', s_2') to itself.

Also

$$\lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{t} p^{(k)}(s'|s)=f_{(s)}^{(s')}\lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{t} p_t^{(k)}(s'), \quad s=(s_1,s_2),\ s'=(s_1',s_2').$$

Since the Markov Chain $\{(X_t, I_t): t=0,1,2,\ldots\}$ is a unichain which is irreducible, all its states are ergodic and have a unique equilibrium distribution.

Thus, $\pi_{(s')}(R)=\lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{t} p^{(k)}(s'|s)(R), \quad s=(s_1,s_2), \ s'=(s_1',s_2'),$

exist and is independent of initial state , such that $\pi P=\pi$ and $\sum_{s\in S}\pi_{(s)}=1$.

VI. OPTIMAL POLICY

A stationary policy $R*$ is said to be an average cost optimal policy if $h_{s_1,s_2}(R*)\le h_{s_1,s_2}(R)$ for each stationary policy R uniformly with the initial state (s_1, s_2).

The relative value associated with a given policy R provides a tool for constructing a new policy $R*$ whose average cost is no more than that of the current policy R.

The objective is to improve the given policy R whose average cost is $h(R)$ and relative value $v_{(s_1,s_2)}(R), \ (s_1,s_2)\in S$.

By constructing a new policy R such that for each $(s_1,s_2)\in S$,

$$c_{(s)}(R*)-h(R)+\sum_{s'\in S} p_{(s,s')}(R*)v_{s'}\le v_s\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where, $s=(s_1,s_2)$ and $s'=(s_1',s_2')$, we obtain an improved rule $R*$ with $h(R*)\le h(R)$. We have to find the optimal policy $R*$ satisfying (1) which minimizes the cost functions $c_i(a)-h(R)+\sum_{s'\in S} p_t(s'|s,a)v_{s'}(R)$ over all actions $a\in A(s)$.

VII. ALGORITHM

Step 0: (Initialization)

Choose a stationary policy R for the periodic review based admission control in service facility system with two types of customers.

Step 1: (Value determination step)

For the current policy R, compute the unique solution $(h(R), v_s(R))$ to the following linear equations

$$v_s=c_s(R)-h(R)+\sum_{s'\in S} p_t(s'|s)(R)v_{s'}(R), \quad s=(s_1,s_2)\in S,$$

$$v_s=0, \quad where\ s=(s_1,s_2)\ is\ arbitarily\ chosen\ state\ in\ S.$$

Step 2:(Policy Improvement)

For each state $s=(s_1,s_2)\in S$ determine the actions yielding, optimal cost, that is

$$a^*\in\arg\min_{a\in A_s}\left\{c_s(a)-h(R)+\sum_{s'\in S} p_t(s'|s,a)v_{s'}(R)\right\}.$$

The new stationary policy R^* is obtained by choosing $R^* = a_s$.

Step 3:(Convergence test)

If the new policy $R^* = R$ (the old one), then the searching process stops with policy R. Otherwise go to **Step 1** with R replaced by new R^*.

VIII. NUMERICAL EXAMPLE

Consider a MDP formulation of a service facility system with two types of customers. Admission to the system is controlled by observing the number of customers in the eligible queue and potential queue. Decisions at equidistant time epochs are taken to admit the eligible number of customers by observing the different category of customers and available empty space in the system.

For the system we assume, $N = 5$. Let $\{X(t) : t \geq 0\}$ where X(t) denote the number of customers in the eligible queue be a stochastic process with has state space $S_1 = \{0,1,2,3,4,5\}$ and action set $A = \{0,1,2,\ldots,s_2\}$, where $s_1 + s_2 \leq 5$.

Assume that the incentive cost $c_{r1} = 0.1$ per customer for priority and waiting cost be $c_w = 0.01$ per customer.

$s_1 \setminus s_1^{'}$	**5**	**4**	**3**	**2**	**1**	**0**
5	0.27	0.35	0.20	0.10	0.05	0.03
4	0.01	0.20	0.37	0.30	0.10	0.02
3	0.02	0.04	0.24	0.5	0.15	0.05
2	0.02	0.04	0.05	0.24	0.55	0.10
1	0.01	0.03	0.04	0.05	0.37	0.50
0	0.02	0.03	0.05	0.09	0.12	0.69

Computational Procedure:

For any given policy R, the policy improvement quantity is given by

$$T_s(a,R) = c_s(a) - h(R) + \sum_{s' \in S} p_t(s'|s,a) v_{s'}(a) \quad where \; T_s(a,R) = v_s(R) \quad for \; a = R_s.$$

Iteration 1:

Policy iteration algorithm is initialized with $R^{(1)} = (0,0,0,0,0,5)$, which prescribes the transfer of 5 priority customers from potential queue to the system(eligible queue + 1 in server) when there is no customer in the system. Solving the system of linear equations connecting the average cost $h(R)^{(1)}$ by assuming $v_5 = 0$ we get

$v_5(R^{(1)}) = 0$, $v_4(R^{(1)}) = 0.1424144091$, $v_3(R^{(1)}) = 0.2601353793$, $v_2(R^{(1)}) = 0.4703303818$, $v_1(R^{(1)}) = 0.7836061876$, $v_0(R^{(1)}) = 1.369282975$, $h(R^{(1)}) = 0.2291639559$

$T_s(a, R^{(1)})$					
$s_1 \setminus a$	4	3	2	1	0
4	X	X	x	**<u>0.15</u>**	0.1824144090
3	X	X	0.25	**<u>0.14</u>**	0.2901353793
2	X	0.35	0.24	**<u>0.13</u>**	0.4903303818
1	0.45	0.34	0.23	**<u>0.12</u>**	0.7936061876

The new policy will be $R^{(2)} = (0,1,1,1,1,5)$. Since the new policy $R^{(2)}$ is different from the initial policy $R^{(1)}$ the searching process continues.

Iteration 2:

For the policy $R^{(2)}$, solving the system of linear equations connecting the average cost $h(R)^{(2)}$ by assuming $v_5 = 0$ we get

$v_5\left(R^{(2)}\right) = 0$, $v_4\left(R^{(2)}\right) = 0.2484449811$, $v_3\left(R^{(2)}\right) = 0.3411719874$, $v_2\left(R^{(2)}\right) = 0.5096793582$, $v_1\left(R^{(2)}\right) =$ 0.7611227312, $v_0\left(R^{(2)}\right) = 1.227951781$, $h\left(R^{(2)}\right) = 0.2810527667$

$T_s\left(a,\ R^{(2)}\right)$					
$s_1 \backslash a$	4	3	2	1	0
4	X	X	X	**0.15**	0.1884449810
3	X	X	0.25	**0.14**	0.2711719874
2	X	0.35	0.24	**0.13**	0.4296793582
1	0.45	0.34	0.23	**0.12**	0.6711227311

Since the new policy $R^{(3)} = (0,1,1,1,1,5)$ is identical with the policy, the searching process stops here. After two iterations we obtained the optimal policy $R^{*} = (0,1,1,1,1,5)$ which prescribes the following rule: It is beneficial to allow no customer, 1 priority customer, 1 priority and 0 or 1 ordinary customer, 1 priority and 0 or 1 or 2 ordinary customers, 1 priority and 0 or 1 or 2 or 3 ordinary customers and 5 priority customers to the system at system states: 5, 4, 3, 2, 1and 0 respectively.

IX. CONCLUSIONS

The In this article we analyzed a discrete time MDP in service facility systems with two types of customers. We control the number of customers admitted to the system by observing two types of customers in the potential queue and empty space in the system. Decision to admit customers is made at the beginning of each period. In future we would like to extend the model to control both service and inventory in a service facility with inventory management.

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