Best proximity point theorems for generalized JSC- proximal contractions

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Abstract

This article presents best proximity point theorems for new classes of non-self mappings, known as generalized JSC-proximal contractions in metric spaces.Presented results and theorems are generalizations of [8] and [9]

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1 Introduction

Fixed point theory focusses on the strategies for solving non-linear equations of the kind $Tx = x$ in which T is a self mapping defined on a subset of a metric space, a normed linear space, a topological vector space or some pertinent framework. But, when T is not a self-mapping, it is plausible that $Tx = x$ has no solution.

If A and B are two non-empty subsets of a metric space (X, d), then an element $x \in A$ is said to be a fixed point of a given map $T : A \to B$ if $Tx = x$. Clearly, $T(A) \cap A \neq \emptyset$ ia a necessary(but not sufficient) condition for the existence of a fixed point of T. If $T(A) \cap A = \emptyset$, then $d(x, Tx) > 0$ for all $x \in A$, that is, the set of fixed points of T is empty. In such a situation, one often attempts to find an element x which is in some sense closest to Tx. Best proximity point analysis has been developed in this direction.

An element $x \in A$ is called a best proximity of T if $d(x, Tx) = d(A, B)$. Indeed, a best proximity point theorem details sufficient conditions for the existence of an element x such that the error $d(x, Tx)$ is minimum. A best proximity point theorem is fundamentally concerned with the global minimization of the real valued function $x \to d(x, Tx)$ that is an indicator of the error involved for an approximate solution of the equation $Tx = x$. Because of the fact that, for a non-self mapping $T : A \to B$, $d(x, Tx)$ is at least $d(A, B)$ for all $x \in A$, a best proximity point theorem ensures global minimum of the error $d(x, Tx)$ by confining an approximate solution x of the equation $Tx = x$ to comply with the condition that $d(x, Tx) = d(A, B)$. Such an optimal approximate solution of the equation $Tx = x$ is said to be a best proximity point of the non-self mapping $T : A \to B$.

The primary objective of this article is to be provide best proximity point theorems for generalized JSC- proximal contractions of the first and the second kinds in the setting of complete metric spaces, thereby ascertaining an optimal approximate solution to the equation $Tx = x$, where $T : A \to B$ is a generalized JSC-proximal contraction of the first or generalized JSC-proximal contraction of the second kind.

2 Preliminaries

Let A and B be the non-empty subsets of a metric space. We know that the following notations and notions are used in the sequel.
$d(A, B) = \inf\{d(x, y) : x \in A \quad \text{and} \quad y \in B\}$
$A_0 = \{x \in A : d(x, y) = d(A, B) \quad \text{for some} \quad y \in B\}$
$B_0 = \{y \in B : d(x, y) = d(A, B) \quad \text{for some} \quad x \in A\}$
If A and B are closed subsets of a normed linear space such that $d(A, B) > 0$, then A_0 and B_0 are contained in the boundaries of A and B respectively [7].

Definition 2.1. *[9] The Set B is said to be approximately compact with respect to A if every sequence $\{y_n\}$ of B satisfying the condition that $d(x, y_n) \to$*

$d(x, B)$ for some $x \in A$ has a convergent subsequence.

Definition 2.2. *A is said to be approximately compact with respect to B if every sequence $\{x_n\}$ of A satisfying the condition that $d(y, x_n) \to d(y, A)$ for some y in B has a convergent subsequence.*

It is obvious that any compact set is approximately compact,and that any set is approximately compact with respect to itself. Further, if A is compact and B is approximately compact with respect to A, then it is ensured that A_0 an B_0 are non-empty.

Definition 2.3. *[8] Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a JSC-contraction if there exist $\psi \in \Psi$ and non-negative numbers q, r, s, t with $q + r + s + 2t < 1$ such that*

$$\psi(d(Tx, Ty)) \leq q\psi(d(x, y))+r\psi(d(x, Tx))+s\psi(d(y, Ty))+t\psi(d(x, Ty)+d(y, Tx)), \forall x, y \in X.$$

where Ψ is the set of all functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying conditions:
(ψ_1) ψ is non-decreasing and $\psi(t) = 0$ if and only if $t = 0$;
(ψ_2) for each sequence $\{t_n\} \subset (0, +\infty)$, $\lim_{n\to\infty} \psi(t_n) = 0$
if and only if $\lim_{n\to\infty} t_n = 0$;
(ψ_3) there exist $r \in (0, 1)$ and $l \in (0, +\infty]$ such that $\lim_{t\to 0^+} \dfrac{\psi(t)}{t^r} = l$;
(ψ_4) $\psi(a + b) \leq \psi(a) + \psi(b)$ for all $a, b > 0$.

For convenience, we denote by Ψ_1, the set of all non-decreasing functions ψ:$(0, +\infty) \to (0, +\infty)$ satisfying (ψ_2) and (ψ_3) and by (Ψ_2), the set of all functions ψ:$[0, +\infty) \to [0, +\infty)$ satisfying (ψ_1), (ψ_2) and (ψ_4).

3 Main Results

Definition 3.1. *A mapping $T : A \to B$ is said to be generalized JSC-proximal contraction of the first kind if there exists $\psi \in \Psi$ and non-negative numbers q, r, s, t with $q + r + s + 2t < 1$ such that the conditions*
$d(u_1, Tx_1) = d(A, B)$ and $d(u_2, Tx_2) = d(A, B)$

implies the inequality

$$\psi(d(u_1,u_2)) \leq q\psi(d(x_1,x_2)) + r\psi(d(x_1,u_1)) + s\psi(d(x_2,u_2)) + t\psi(d(x_1,u_2)+d(x_2,u_1))$$

for all $u_1, u_2, x_1, x_2 \in A$. *If* T *is a self mapping on* A, *then the preceding definition reduces to the condition that*

$$\psi(d(Tx_1,Tx_2)) \leq q\psi(d(x_1,x_2)) + r\psi(d(x_1,Tx_1)) + s\psi(d(x_2,Tx_2)) + t\psi(d(x_1,Tx_2)+d(x_2,Tx_1))$$

for all $x_1, x_2 \in A$. *where* Ψ *is the set of all functions* $\psi : [0,+\infty) \to [0,+\infty)$ *satisfying conditions:*
(ψ_1) ψ *is non-decreasing and* $\psi(t) = 0$ *if and only if* $t = 0$*;*
(ψ_2) *for each sequence* $\{t_n\} \subset (0,+\infty)$, $\lim\limits_{n\to\infty} \psi(t_n) = 0$
if and only if $\lim\limits_{n\to\infty} t_n = 0$*;*
(ψ_3) *there exist* $r \in (0,1)$ *and* $l \in (0,+\infty]$ *such that* $\lim\limits_{t\to 0^+} \dfrac{\psi(t)}{t^r} = l$*;*
(ψ_4) $\psi(a+b) \leq \psi(a) + \psi(b)$ *for all* $a, b > 0$.

Definition 3.2. *A mapping* $T : A \to B$ *is said to be generalized JSC-proximal contraction of the second kind if there exists* $\psi \in \Psi$ *and non-negative numbers* q, r, s, t *with* $q + r + s + 2t < 1$ *such that the conditions*
$d(u_1, Tx_1) = d(A, B)$ *and* $d(u_2, Tx_2) = d(A, B)$
imply that

$$\psi(d(Tu_1,Tu_2)) \leq q\psi(d(Tx_1,Tx_2)) + r\psi(d(Tx_1,Tu_1)) + s\psi(d(Tx_2,Tu_2)) + t\psi(d(Tx_1,Tu_2)+d(Tx_2,Tu_1))$$

for all $u_1, u_2, x_1, x_2 \in A$.

4 Generalized JSC-proximal contraction

The following main result is a best proximity point theorem for non-self generalized JSC proximal contractions of the first kind, that are not necessarily continuous.

Theorem 4.1. *Let* A *and* B *be non-empty closed subsets of a complete metric space such that* B *is approximately compact with respect to* A. *Moreover,* *If A_0 and B_0 are non-empty. Let $T : A \to B$ satisfy the following conditions:*

(a) T is generalized JSC-proximal contraction of the first kind.

(b) $T(A_0)$ is contained in B_0.

Then, there exists a unique element x in A such that $d(x, Tx) = d(A, B)$ Also, for any fixed element $x_0 \in A_0$, the sequence $\{x_n\}$ defined by $d(x_{n+1}, Tx_n) = d(A, B)$ converges to the best proximity point x.

Proof. Let $x_0 \in A_0$.
Since $T(A_0)$ is contained in B_0, this implies that there is an element x_1 in A_0 satisfies $d(x_1, Tx_0) = d(A, B)$
Also, since $Tx_1 \in T(A_0)$ which is contained in B_0, it follows that there is an element x_2 in A_0 such that $d(x_2, Tx_1) = d(A, B)$
Continuing this process, choose x_n in A_0, then there is an element x_{n+1} in A_0 satisfying the condition that $d(x_{n+1}, Tx_n) = d(A, B)$
for every non-negative integer n because of the fact that $T(A_0)$ is contained in B_0.
Since T is a generalized JSC-proximal contraction of the first kind, we have

$$\begin{aligned}
\psi(d(x_n, x_{n+1})) &\leq q\psi(d(x_{n-1}, x_n)) + r\psi(d(x_{n-1}, x_n)) + s\psi(d(x_n, x_{n+1})) + t\psi(d(x_{n-1}, x_{n+1})) \\
&\leq q\psi(d(x_{n-1}, x_n)) + r\psi(d(x_{n-1}, x_n)) + s\psi(d(x_n, x_{n+1}))+ \\
&\quad t\psi(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \\
&\leq q\psi(d(x_{n-1}, x_n)) + r\psi(d(x_{n-1}, x_n)) + s\psi(d(x_n, x_{n+1}))+ \\
&\quad t\psi(d(x_{n-1}, x_n)) + t\psi(d(x_n, x_{n+1})) \\
&= \psi(d(x_{n-1}, x_n))[q + r + t] + \psi(d(x_n, x_{n+1}))[s + t] \\
\psi(d(x_n, x_{n+1})) &\leq \frac{q + r + t}{1 - s - t}\psi(d(x_{n-1}, x_n))
\end{aligned}$$

This implies
$\psi(d(x_n, x_{n+1})) \leq k\psi(d(x_{n-1}, x_n))$
where $k = \dfrac{q + r + t}{1 - s - t}$ which is strictly less than 1.
Therefore, $\{x_n\}$ is a Cauchy sequence. Since the space is complete, the sequence $\{x_n\}$ converges to some element x in A.
Moreover,

$$\begin{aligned} d(x, B) \leq d(x, Tx_n) &\leq d(x, x_{n+1}) + d(x_{n+1}, Tx_n) \\ &= d(x, x_{n+1}) + d(A, B) \\ &= d(x, x_{n+1}) + d(x, B) \end{aligned}$$

Therefore $d(x, Tx_n) \to d(x, B)$
Since B is approximately compact with respect to A, the sequence $\{Tx_n\}$ has a subsequence $\{Tx_{n_k}\}$ converging to some element y in B. We have
$d(x, y) = \lim_{n\to\infty} d(x_{n_k+1}, Tx_{n_k}) = d(A, B)$
This implies that x must be an element of A_0. Because $T(A_0)$ is contained in B_0, we have
$d(u, Tx) = d(A, B)$ for some $u \in A$.
Since T is a generalized JSC-proximal contraction of the first kind, we have

$$\begin{aligned} \psi(d(u, x_{n+1})) \leq\, & q\psi(d(x, x_n)) + r\psi(d(u, x)) + s\psi(d(x_n, x_{n+1}))+ \\ & t\psi(d(x, x_{n+1}) + d(x_n, u)) \end{aligned}$$

Letting $n \to \infty$

$$\begin{aligned} \psi(d(u, x)) &\leq q\psi(d(x, x)) + r\psi(d(u, x)) + s\psi(d(x, x))+ \\ &\quad t\psi(d(x, x) + d(x, u)) \\ &= r\psi(d(u, x)) + t\psi(d(x, u)) \\ &= (r + t)\psi(d(x, u)) \end{aligned}$$

This implies x and u must be identical. Thus, it follows that
$d(x, Tx) = d(u, Tx) = d(A, B)$. Therefore x is a best proximity point.
Now to prove the uniqueness of best proximity point.
Suppose that there is another best proximity point x^* of T, so that $d(x^*, Tx^*) = d(A, B)$
Since T is generalized JSC-proximal contraction of the first kind, we have
$\psi(d(x, x^*)) \leq (q + 2t)\psi(d(x, x^*))$
This implies x and x^* are identical.
Hence T has a unique best proximity point. This completes the proof.

∎

The previous theorem contains the following result which gives Banach's contraction principle by defining $\psi(t) = t$

Corollary 4.1. *Let A and B be non-empty, closed subsets of a complete metric space such that B is approximately compact with respect to A. Also assume that A_0 and B_0 are non-empty. Let $T : A \to$ satisfy the following conditions:*

(a) *There exist* $\psi \in \Psi_2$ *with* $\psi(t) = t$ *and non-negative real number* $k < 1$ *such that, for all* u_1, u_2, x_1, x_2 *in* A,
$d(u_1, Tx_1) = d(A, B)$ *and* $d(u_2, Tx_2) = d(A, B)$
$\Rightarrow \psi(d(u_1, u_2)) \leq k\psi(d(x_1, x_2))$.

(b) $T(A_0) \subseteq B_0$

Then there exists a unique element $x \in A$ *such that*
$d(x, Tx) = d(A, B)$
Further, for any fixed element $x_0 \in A_0$, *the sequence* $\{x_n\}$, *defined by*
$d(x_{n+1}, Tx_n) = d(A, B)$,
converges to the best proximity point x.

The best proximity point Theorem4.1 includes the following fixed point theorem as a special case[8].

Corollary 4.2. *Let* (X, d) *be a complete metric space. Let* T *be a self mapping on* X. *Assume that there exits* $\psi \in \Psi_2$ *ans non-negative real numbers* q, r, s, t *with* $q + r + s + 2t < 1$ *such that*

$$\psi(d(Tx_1, Tx_2)) \leq q\psi(d(x_1, x_2)) + r\psi(d(x_1, Tx_1)) + s\psi(d(x_2, Tx_2)) + t\psi[d(x_1, Tx_2) + d(x_2, Tx_1)]$$

for all x_1, x_2 *in the domain of the mapping* T.
Then T *has a unique fixed point.*

The following main result is a best proximity point theorem for non-self generalized JSC-proximal contractions of the second kind.

Theorem 4.2. *Let* A *and* B *be non-empty, closed subsets of a complete metric space such that* A *is approximately compact with respect to* B. *Also, suppose that* A_0 *and* B_0 *are non-empty. Let* $T : A \to B$ *satisfy the following conditions:*

(a) *T is a continuous generalized JSC-proximal contraction of the second kind.*

(b) $T(A_0)$ is contained in B_0.

Then, there exists an element $x \in A$ such that
$d(x, Tx) = d(A, B)$
and the sequence $\{x_n\}$ converges to the best proximity point x,where x_0 is any fixed element in A_0 and $d(x_{n+1}, Tx_n) = d(A, B)$, for $n \geq 0$.

Further, if x^* is another best proximity point of T, then $Tx = Tx^*$, hence T is a constant on the set of all best proximity points of T.

Proof. Choose an element x_0 in A_0. On account of the fact $T(A_0)$ is contained in B_0, it is guaranteed that there is an element x_1 in A_0 satisfying the condition that $d(x_1, Tx_0) = d(A, B)$.
Since Tx_1 is an element of $T(A_0)$ which is contained in B_0, then there exists an element $x_2 \in A_0$ such that
$d(x_2, Tx_1) = d(A, B)$
Continuing this process, we obtain the sequence $\{x_n\}$ in A_0 such that $d(x_{n+1}, Tx_n) = d(A, B)$, where $x_n, x_{n+1} \in A_0$ for all non-negative integral values of n.
Since T is a generalized JSC-proximal contraction of the second kind,

$$\begin{aligned}
\psi(d(Tx_n, Tx_{n+1})) &\leq q\psi(d(Tx_{n-1}, Tx_n)) + r\psi(d(Tx_{n-1}, Tx_n)) + s\psi(d(Tx_n, Tx_{n+1})) \\
&\quad + t\psi(d(Tx_{n-1}, Tx_{n+1})) \\
&\leq q\psi(d(Tx_{n-1}, Tx_n)) + r\psi(d(Tx_{n-1}, Tx_n)) + s\psi(d(Tx_n, Tx_{n+1})) + \\
&\quad t\psi(d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})) \\
&= \psi(d(Tx_{n-1}, Tx_n))[q + r + t] + \psi(d(Tx_n, Tx_{n+1}))[s + t] \\
\psi(d(Tx_n, Tx_{n+1})) &\leq \frac{q + r + t}{1 - s - t}\psi(d(Tx_{n-1}, Tx_n))
\end{aligned}$$

This implies
$\psi(d(Tx_n, Tx_{n+1})) \leq k\psi(d(Tx_{n-1}, Tx_n))$
where $k = \dfrac{q + r + t}{1 - s - t}$ which is strictly less than 1.
Therefore, $\{Tx_n\}$ is a Cauchy sequence. Since the space is complete, the sequence $\{Tx_n\}$ converges to some element y in B.
Moreover,

$$\begin{aligned}
d(y, A) \leq d(y, x_{n+1}) &\leq d(y, Tx_n) + d(Tx_n, x_{n+1}) \\
&= d(y, Tx_n) + d(B, A) \\
&= d(y, Tx_n) + d(y, A)
\end{aligned}$$

Therefore $d(y, x_n) \to d(x, B)$
Since A is approximately compact with respect to B, the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to some element x in A. since T is a continuous map,
$d(x, Tx) = \lim_{n\to\infty} d(x_{n+1}, Tx_n) = d(A, B)$
Therefore x is the best proximity point in A.
Suppose that there is another best proximity point x^* in A so that
$d(x^*, Tx^*) = d(A, B)$
Because T is a generalized JSC-proximal contraction of the second kind,
$\psi(d(Tx, Tx^*)) \leq (q + 2t)\psi(d(Tx, Tx^*))$
This implies that $Tx = Tx^*$. This completes the proof. ∎

Corollary 4.3. *Let A and B be non-empty, closed subsets of a complete metric space such that A is approximately compact with respect to B. Also assume that A_0 and B_0 are non-empty. Let $T : A \to B$ satisfy the following conditions:*

(a) There exists $\psi \in \Psi_2$ with $\psi(t) = t$ and non-negative real number $k < 1$ such that for all u_1, u_2, x_1, x_2 in A
$d(u_1, Tx_1) = d(A, B)$ and $d(u_2, Tx_2) = d(A, B)$
$\Rightarrow \psi(d(Tu_1, Tu_2)) \leq k\psi(d(Tx_1, Tx_2))$

(b) T is continuous.

(c) $T(A_0) \subseteq B_0$.

Then, there exists an unique element $x \in A$ such that
$d(x, Tx) = d(A, B)$
Further, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by
$d(x_{n+1}, Tx_n) = d(A, B)$ converges to a best proximity point of T.

Theorem 4.3. *Let A and B be non-empty, closed subsets of a complete metric space. Also, assume that A_0 and B_0 are non-empty. Let $T : A \to B$ satisfy the following conditions:*

(a) T is a generalized JSC-proximal contraction of the first kind as well as a generalized JSC-proximal contraction of the second kind.

(b) $T(A_0)$ is contained in B_0.

Then, there exists a unique element x in A such that
$d(x, Tx) = d(A, B)$,
and the sequence $\{x_n\}$ converges to the best proximity point x, where x_0 is any fixed element in A_0 and $d(x_{n+1}, Tx_n) = d(A, B)$ for $n \geq 0$.

Proof. Proceeding as in Theorem4.1, it is possible to find a sequence $\{x_n\}$ in A_0 such that
$d(x_{n+1}, Tx_n) = d(A, B)$
for all no-negative integral values of n.
As in Theorem4.1, we can prove that the sequence $\{x_n\}$ is a Cauchy sequence and hence converges to some x in A.
As in Theorem4.2, it can be shown that the sequence $\{Tx_n\}$ is a Cauchy sequence and converges to some y in B.
Therefore, we have
$d(x, y) = \lim_{n\to B} d(x_{n+1}, Tx_n) = d(A, B)$
This implies that x becomes an element of A_0. Since $T(A_0)$ is contained in B_0, we have
$d(u, Tx) = d(A, B)$
for some element $u \in A$.
Since T is a generalized JSC-proximal contraction of the first kind,it can be seen that

$$\begin{aligned}\psi(d(u, x_{n+1})) &\leq q\psi(d(x, x_n)) + r\psi(d(u, x)) + s\psi(d(x_n, x_{n+1})) \\ &\quad + t\psi[d(x, x_{n+1}) + d(x_n, u)]\end{aligned}$$

as $\quad n \to \infty$, we get

$$\begin{aligned}d(u, x) &\leq r\psi(d(u, x)) + t\psi(d(x, u)) \\ &= (r + t)\psi(d(u, x))\end{aligned}$$

This implies that x and u are identical.
Thus, we have $d(x, Tx) = d(u, Tx) = d(A, B)$
Now to show that the uniqueness of the best proximity point. Suppose that there is another best proximity point x^* of T, we have
$d(x^*, Tx^*) = d(A, B)$
Since T is a generalized JSC-proximal contraction of the first kind,
$\psi(d(x, x^*)) \leq (q + 2t)\psi(d(x, x^*))$
This implies $x = x^*$
Hence T has a unique best proximity point.This completes the proof. ∎

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