Four-Step Block Method for Solving Third Order Ordinary Differential Equation

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Abstract - *In This paper we proposed a Four-Step Block- method of Hybrid Linear Multistep Method derived by collocation and interpolation technique with power series as basis function for direct solution of special third order initial value problem of ordinary differential equation. The basic properties of the block, which include consistency, Zero Stable, order, error constant and region of absolute stability were analysed. From the analysis the block derived was found to be Consistent, Zero stable, convergent and A-stable. Also, the block- method was tested on some numerical examples and the result computed shows the derived block method is more accurate than some existing methods considered in this paper.*

Keywords – *Zero Stable, Error constant, Region of Absolute stability and Convergence.*

I. INTRODUCTION

This paper considered the development of a 4- Step Block Hybrid Linear Multistep method for the direct solution of special third order Initial value problem of ordinary differential equation(ODE) of the form,

$$y''' = f(x, y), y(x_0) = y_0 \ , \ y'(x_0) = y'_0 \, , y''(x_0) = y''_0 \ \ y \in \Re \ . \qquad (1)$$

Third order differential equations usually arise in many physical problems such as electromagnetic waves, thin film flow and gravity- driven flows. Therefore, third order ODEs have attracted considerable attention. Many theoretical and numerical studies dealing with this type of equations have been discussed in literature. The popular approach for solving this type of equations is by converting it to a system of first order ODE, and solving it using any suitable method in literature.

Recent research in this area includes [3], a P-stable linear multistep method for general third order initial value problem of ODE, which is an improvement over Nystrom's method, but the method was implemented in predictor-corrector mode. Like other linear multistep methods they have their draw backs, which includes not being self-starting, secondly, they advance the numerical integration of ODE in one-step at a time. [1] developed a hybrid formula of order four for starting the Numerov method applied to the initial value problem for solving special second order ODE. [10] derived a new multiple fnite difference methods through multistep collocation for special second order ODE. [8] developed a P- Stable block linear multistep method for solving third order ODE using collocation and interpolation technique on power series approximation method. The schemes were applied as a simultaneous integrator, and the derived block method possessed the desirable features of Runge-Kutta methods of being self-starting, that is eliminating the use of predictors, which is an improvement over [3].

This paper therefore proposes development of Block method that will be highly efficient in terms of accuracy, small error constant, possess better rate of convergence , and very easy to implement.

Definition 1: Initial Value Problem
This is any differential equation in which the initial condition of the problem is given, it is given in the form

$$y^k(x) = f(x, y, y' ... y^{k-1}) \ , \ y(x_0) = y_0 \, , \ y'(x_0) = y'_0 \ ... y^{k-1}(x_0) = y_0^{k-1} \qquad (2)$$

$a \le x \le b,$ given $a = x_0 < x_1 ... < N = b$

Definition 2: Block Methods

A block method is formulated in Linear Multistep method form. According to Lambert in [2], block method preserve the advantage of one step methods, of being self- starting and permitting easy change of step length.

According to [2] a general k- block, r-point block method is a matrix of finite difference equation of the form

$$Y_m = \sum_{j=1}^{k} A_j y_{m-j} + h\sum_{j=0}^{k} B_j F_{m-j} \tag{2b}$$

where $Y_m = \left(y_n, y_{n+1}, ..., y_{n+r-1}\right)^T, F_m = \left(f_n, f_{n+1}, ..., f_{n+r-1}\right)^T$ A_j 's and B_j 's are properly chosen r $\times$ r matrix coefficients and m= 0,1,2…represents the number $n = mr$ is the first step number of the mth block and r is the proposed block size.

Definition 3: Consistency [7]

Linear Multistep Methods are said to be consistent if it has order $p \geq 1$.

Definition 4:Convergence [7]

The necessary and sufficient conditions that a Linear Multistep Method be convergent are consistency and zero-stable.

Definition 5: Zero stability [7]

A block method is said to be zero-stable if the roots $\lambda_i, i = 1,2,3...s$ of the characteristic polynomial $\rho(\lambda)$ defined by $\rho(\lambda) = \left|\sum_{i=0}^{s} A^i \lambda^{s-i}\right| = 0$ satisfies $|\lambda_i| \leq 1$ and for those roots with $|\lambda_i| = 1$, the multiplicity must not exceed the order of the differential equation in consideration.

Definition 6: Region of Absolute Stability

Hairer and Wanner in [9] defined the region of absolute stability as the set of W equals $\tau \in \square$; all the roots $\xi_i(\tau)$ of the characteristics equation satisfying $|\xi_i(\tau)| < 1$ is called the stability region or region of absolute stability of the method.

Definition 7: A-Stability [7]

An A-stable method is a numerical method in which the stability region contains the entire left half plane.

Definition 7: Order and Error constant

A linear multistep method of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^3 \sum_{j=0}^{k} \beta_j f_{n+j} \tag{3}$$

$k \geq 3$ is said to be of order P if $C_0 = C_1 = C_2 = ... = C_P = C_{P+1} = C_{P+2} = 0$ but $C_{p+3} \neq 0$, C_{p+3} is called the error constant and $C_{p+3} h^{p+3} y^{(p+3)}(x_n)$ is the principal local truncated error at the point x_n .The local truncated error associated with the equation (27) was defined by a linear difference operator L as

$$L[y(x);h] = C_0 y(x) + C_1 h y'(x) + ... + C_q h^q y^q(x) + ...$$

Where $q = 3, 4, \ldots$ were given as

$$C_0 = \sum_{j=0}^{k} \alpha_j$$

$$C_1 = \sum_{j=1}^{k} j\alpha_j$$

$$C_2 = \sum_{j=1}^{k} j^2\alpha_j$$

$$C_3 = \frac{1}{3!}\left(\sum_{j=1}^{k} j^3\alpha_j\right) - \left(\sum_{j=1}^{k} \beta_j\right)$$

$$C_4 = \frac{1}{4!}\left(\sum_{j=1}^{k} j^4\alpha_j\right) - \left(\sum_{j=1}^{k} j\beta_j\right)$$

$$C_5 = \frac{1}{5!}\left(\sum_{j=1}^{k} j^5\alpha_j\right) - \left(\frac{1}{2!}\sum_{j=1}^{k} j^2\beta_j\right)$$

(29)

$$C_q = \frac{1}{q!}\left(\sum_{j=1}^{k} j^q\alpha_j\right) - \left(\frac{1}{(q-3)!}\sum_{j=1}^{k} j^{q-3}\beta_j\right)$$

II. METHODOLOGY

A power series of a single variable x in the form:

$$P(x) = \sum_{j=0}^{\infty} a_j x^j \tag{4}$$

is used as the basis or trial function, to produce the approximate solution given as

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j \qquad a_j \in R, y \in C^M(a,b) \subset \mathrm{P(X)} \tag{5}$$

where s and r are the number of collocation and interpolation points respectively.
The derivatives are given as follows

$$y'(x) = \sum_{j=1}^{r+s-1} j a_j x^{j-1}, \tag{6}$$

$$y''(x) = \sum_{j=2}^{r+s-1} j(j-1) a_j x^{j-2}, \tag{7}$$

$$y'''(x) = \sum_{j=3}^{r+s-1} j(j-1)(j-2) a_j x^{j-3}. \tag{8}$$

Substituting (8) into (1) gives

$$f(x,y)=\sum_{j=3}^{r+s-1} j(j-1)(j-2)a_j x^{j-3}, \quad (9)$$

where a_j, are the parameters to be determined.

Collocating (9) at both step and off-step points $x=x_{n+p}$, $p=0,v...k$, where k is number of steps , v are the off-step points used in the collocation. Then interpolating (5) at both step and off -step points $x=x_{n+q}$, $q=0,u,w$ where u and w are the off-step points used in the interpolation, Then gives a systems of nonlinear equation in the form

$$AX=U \quad (10)$$

which is solved to obtain the values of parameters a_j's $j=0,1...,(r+s-1)$,that are then substituted into (5), which after some manipulation, yields the new continuous method expressed in the form

$$y(x)=\alpha_0(x)y_n+\alpha_u(x)y_{n+u}+\alpha_w(x)y_{n+w}+h^3\left(\sum_{j=0}^{k}\beta_j f_{n+j}+\sum\beta_v f_{n+v}\right) \quad (11)$$

We express the coefficients (α's and β's) of the continuous scheme as continuous function of t by letting

$$t=\frac{x-x_{n+u}}{h} \quad (12)$$

and noting that

$$\frac{dt}{dx}=\frac{1}{h} \text{ and } \frac{d^2t}{dx^2}=\frac{1}{h^2}$$

Derivation of Four Steps Method with four Off- Step Points at Collocation and two Off- Step Points at Interpolation

For this method we considered three interpolation points and nine collocation points, we have $u=\frac{1}{3}$, $w=\frac{2}{3}$, $v=\frac{1}{3},\frac{2}{3},\frac{4}{3}$ and $\frac{5}{3}$ From (8) for $r=3$ and $s=9$ we obtained a polynomial of degree $r+s-1$ as follows;

$$y(x)=\sum_{j=0}^{11} a_j x^j \quad (13)$$

Third derivative is given as

$$y'''(x)=\sum_{j=3}^{11} j(j-1)(j-2)a_j x^{j-3} \quad (14)$$

From (14) we obtain

$$y'''(x)=\sum_{j=3}^{11} j(j-1)(j-2)a_j x^{j-3}=f(x,y) \quad (15)$$

Collocating (15) at x_{n+p} where $p=0,\frac{1}{3},\frac{2}{3},1,\frac{4}{3},\frac{5}{3},2,3$ and interpolating (13) at x_{n+q}, where $q=0,\frac{1}{3}$ and $\frac{2}{3}$ gives a system of equations of the form (10)

$$\begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 & x_n^{10} & x_n^{11} \\
1 & x_{n+1/3} & x_{n+1/3}^2 & x_{n+1/3}^3 & x_{n+1/3}^4 & x_{n+1/3}^5 & x_{n+1/3}^6 & x_{n+1/3}^7 & x_{n+1/3}^8 & x_{n+1/3}^9 & x_{n+1/3}^{10} & x_{n+1/3}^{11} \\
1 & x_{n+2/3} & x_{n+2/3}^2 & x_{n+2/3}^3 & x_{n+2/3}^4 & x_{n+2/3}^5 & x_{n+2/3}^6 & x_{n+2/3}^7 & x_{n+2/3}^8 & x_{n+2/3}^9 & x_{n+2/3}^{10} & x_{n+2/3}^{11} \\
0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & 210x_n^4 & 336x_n^5 & 504x_n^6 & 720x_n^7 & 990x_n^8 \\
0 & 0 & 0 & 6 & 24x_{n+1/3} & 60x_{n+1/3}^2 & 120x_{n+1/3}^3 & 210x_{n+1/3}^4 & 336x_{n+1/3}^5 & 504x_{n+1/3}^6 & 720x_{n+1/3}^7 & 990x_{n+1/3}^8 \\
0 & 0 & 0 & 6 & 24x_{n+2/3} & 60x_{n+2/3}^2 & 120x_{n+2/3}^3 & 210x_{n+2/3}^4 & 336x_{n+2/3}^5 & 504x_{n+2/3}^6 & 720x_{n+2/3}^7 & 990x_{n+2/3}^8 \\
0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & 210x_{n+1}^4 & 336x_{n+1}^5 & 504x_{n+1}^6 & 720x_{n+1}^7 & 990x_{n+1}^8 \\
0 & 0 & 0 & 6 & 24x_{n+4/3} & 60x_{n+4/3}^2 & 120x_{n+4/3}^3 & 210x_{n+4/3}^4 & 336x_{n+4/3}^5 & 504x_{n+4/3}^6 & 720x_{n+4/3}^7 & 990x_{n+4/3}^8 \\
0 & 0 & 0 & 6 & 24x_{n+5/3} & 60x_{n+5/3}^2 & 120x_{n+5/3}^3 & 210x_{n+5/3}^4 & 336x_{n+5/3}^5 & 504x_{n+5/3}^6 & 720x_{n+5/3}^7 & 990x_{n+5/3}^8 \\
0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & 210x_{n+2}^4 & 336x_{n+2}^5 & 504x_{n+2}^6 & 720x_{n+2}^7 & 990x_{n+2}^8 \\
0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 & 210x_{n+3}^4 & 336x_{n+3}^5 & 504x_{n+3}^6 & 720x_{n+3}^7 & 990x_{n+3}^8 \\
0 & 0 & 0 & 6 & 24x_{n+4} & 60x_{n+4}^2 & 120x_{n+4}^3 & 210x_{n+4}^4 & 336x_{n+4}^5 & 504x_{n+4}^6 & 720x_{n+4}^7 & 990x_{n+4}^8
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \\ a_{11} \end{bmatrix}
=
\begin{bmatrix} y_n \\ y_{n+1/3} \\ y_{n+2/3} \\ f_n \\ f_{n+1/3} \\ f_{n+2/3} \\ f_{n+1} \\ f_{n+4/3} \\ f_{n+5/3} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} \tag{16}$$

The coefficient matrix can be solved either by inversion technique or Gaussian elimination technique to obtain the following coefficients of the continuous scheme.

Substituting the coefficients into (11) yield the following continuous scheme

$$\begin{aligned}
y(x) = {} & \left(\frac{9}{2}t^2 - \frac{3}{2}t\right)y_n + \left(1-9t^2\right)y_{n+1/3} + \left(\frac{9}{2}t^2 + \frac{3}{2}t\right)y_{n+2/3} + h^3\Bigg(\bigg(\frac{-92041}{129330432}t + \frac{332053}{117573120}t^2 \\
& - \frac{11}{648}t^4 + \frac{3299}{64800}t^5 - \frac{12563}{172800}t^6 + \frac{12029}{201600}t^7 - \frac{79}{2688}t^8 + \frac{229}{26880}t^9 \\
& - \frac{17}{12800}t^{10} + \frac{3}{35200}t^{11}\bigg)f_n + \bigg(\frac{-11752063}{658627200}t + \frac{263}{15360}t^2 + \frac{1}{6}t^3 \\
& - \frac{1979}{10560}t^4 - \frac{633}{35200}t^5 + \frac{39291}{140800}t^6 - \frac{11403}{35200}t^7 + \frac{5211}{28160}t^8 - \frac{1431}{24640}t^9 \\
& + \frac{243}{25600}t^{10} - \frac{243}{387200s}t^{11}\bigg)f_{n+1/3} + \bigg(\frac{32}{17325}t - \frac{27259}{725760}t^2 + \frac{11}{28}t^4 \\
& - \frac{659}{2800}t^5 - \frac{1401}{3200}t^6 + \frac{8433}{11200}t^7 - \frac{891}{1792}t^8 + \frac{27}{160}t^9 - \frac{81}{2800}t^{10} \\
& + \frac{243}{123200}t^{11}\bigg)f_{n+2/3} + \bigg(\frac{-98281}{26943840}t + \frac{941531}{29393280}t^2 - \frac{55}{162}t^4 + \frac{1319}{3240}t^5 + \\
& + \frac{3271}{8640}t^6 - \frac{4913}{5040}t^7 + \frac{989}{1344}t^8 - \frac{181}{672}t^9 + \frac{31}{640}t^{10} - \frac{3}{880}t^{11}\bigg)f_{n+1} \\
& + \bigg(\frac{690071}{239500800}t - \frac{21277}{967680}t^2 + \frac{11}{48}t^4 - \frac{513}{1600}t^5 - \frac{537}{2560}t^6 + \frac{33921}{44800}t^7 \\
& - \frac{2295}{3584}t^8 + \frac{4509}{17920}t^9 - \frac{243}{5120}t^{10} + \frac{243}{70400}t^{11}\bigg)f_{n+4/3} + \bigg(\frac{-331}{249480}t + \frac{2767}{290304}t^2
\end{aligned}$$

$$-\frac{11}{112}t^4+\frac{1649}{11200}t^5+\frac{489}{6400}t^6-\frac{549}{1600}t^7+\frac{81}{256}t^8-\frac{297}{2240}t^9+\frac{2349}{89600}t^{10}$$

$$-\frac{243}{123200}t^{11}\Big)f_{n+5/3}+\Big(\frac{77291}{269438400}t-\frac{39041}{19595520}t^2+\frac{11}{540}t^4-\frac{343}{10800}t^5$$

$$-\frac{401}{28800}t^6+\frac{2459}{33600}t^7-\frac{641}{8960}t^8+\frac{71}{2240}t^9-\frac{21}{3200}t^{10}+\frac{9}{17600}t^{11}\Big)f_{n+2}$$

$$\Big(\frac{-7333}{1616630400}t+\frac{1009}{33592320}t^2-\frac{11}{36288}t^4+\frac{907}{1814400}t^5+\frac{11}{69120}t^6-$$

$$\frac{227}{201600}t^7+\frac{13}{10752}t^8-\frac{1}{1680}t^9+\frac{1}{7168}t^{10}-\frac{3}{246400}t^{11}\Big)f_{n+3}+$$

$$\Big(\frac{3653}{23710579200}t-\frac{47}{47029248}t^2+\frac{1}{99792}t^4-\frac{169}{9979200}t^5-\frac{17}{3801600}t^6+$$

$$\frac{167}{4435200}t^7-\frac{5}{118272}t^8+\frac{13}{591360}t^9-\frac{1}{179200}t^{10}+\frac{1}{5420800}t^{11}\Big)f_{n+4}\Big) \quad (17)$$

Evaluating (17) at $x=x_{n+1}, x=x_{n+4/3}, x=x_{n+5/3}, x=x_{n+2}, x=x_{n+3}$ and $x=x_{n+4}$ gives value of t to be $\frac{2}{3},1,\frac{4}{3},\frac{5}{3},\frac{8}{3}$ and $\frac{11}{3}$.The following discrete schemes were obtained

$$y_{n+1}-y_n+3y_{n+1/3}-3y_{n+2/3}=h^3\Big(\frac{2027}{6889050}f_n+\frac{683233}{39916800}f_{n+1/3}+\frac{2759}{129600}f_{n+2/3}$$

$$-\frac{65209}{22044960}f_{n+1}+\frac{377}{201600}f_{n+4/3}-\frac{61}{86400}f_{n+5/3}+\frac{9803}{73483200}f_{n+2}-$$

$$\frac{1483}{881798400}f_{n+3}+\frac{247}{489891200}f_{n+4}\Big)$$

The order $P=9$ and the Error constant $C_{12}=-7.054\times10^{-10}$

$$y_{n+4/3}-3y_n+8y_{n+1/3}-6y_{n+2/3}=h^3\Big(\frac{16523}{18370800}f_n+\frac{1541849}{29937600}f_{n+1/3}$$

$$+\frac{36971}{453600}f_{n+2/3}+\frac{20777}{1837080}f_{n+1}+\frac{11153}{2721600}f_{n+4/3}-\frac{31}{21600}f_{n+5/3}+\frac{1109}{4082400}f_{n+2}$$

$$-\frac{251}{73483200}f_{n+3}+\frac{83}{808315200}f_{n+4}\Big)$$

The order $P=9$ and the Error constant $C_{12}=-1.4095\times10^{-9}$

$$y_{n+5/3}-y_n+15y_{n+1/3}-10y_{n+2/3}=h^3\Big(\frac{1763}{979776}f_n+\frac{206191}{1995840}f_{n+1/3}+\frac{701}{3888}f_{n+2/3}$$

$$+\frac{2825}{45927}f_{n+1}+\frac{1027}{40320}f_{n+4/3}-\frac{139}{54432}f_{n+5/3}+\frac{331}{612360}f_{n+2}-\frac{11}{1632960}f_{n+3}$$

$$+\frac{13}{64665216}f_{n+4}\Big)$$

The order $P=9$ and the Error constant $C_{12}=-2.5892\times10^{-9}$

$$y_{n+2}-10y_n+24y_{n+1/3}-15y_{n+2/3}=h^3\Big(\frac{6635}{2204496}f_n+\frac{31279}{181440}f_{n+1/3}+\frac{28877}{90720}f_{n+2/3}$$

$$+\frac{161213}{1102248}f_{n+1}+\frac{15613}{1102248}f_{n+4/3}+\frac{247}{18144}f_{n+5/3}+\frac{8203}{7348320}f_{n+2}$$

$$-\frac{439}{44089920}f_{n+3}+\frac{13}{484989120}f_{n+4}\Big)$$

The order $P=9$ and the Error constant $C_{12}=-3.7153\times10^{-8}$

$$y_{n+3}-28y_n+63y_{n+1/3}-36y_{n+2/3}=h^3\Big(\frac{104933}{5248800}f_n+\frac{106}{275}f_{n+1/3}+\frac{1199}{900}f_{n+2/3}$$

$$-\frac{21827}{131220}f_{n+1}+\frac{65741}{43200}f_{n+4/3}-\frac{3029}{5404}f_{n+5/3}+\frac{249319}{437400}f_{n+2}+\frac{11869}{1312200}f_{n+3}$$

$$+\frac{14131}{115473600}f_{n+4}\Big)$$

The order $P=9$ and the Error constant $C_{12}=-9.2146\times10^{-7}$

$$y_{n+4}-55y_n+120y_{n+1/3}-66y_{n+2/3}=h^3\Big(-\frac{544669}{11022480}f_n+\frac{2894681}{1995840}f_{n+1/3}+\frac{5111}{12960}f_{n+2/3}$$

$$+\frac{4348657}{1102248}f_{n+1}-\frac{45349}{60480}f_{n+4/3}+\frac{30197}{30240}f_{n+5/3}+\frac{12162167}{7348320}f_{n+2}$$

$$+\frac{21871021}{44089920}f_{n+3}+\frac{3718123}{484989120}f_{n+4}\Big)$$

(18)

The order $P=9$ and the Error constant $C_{12}=-1.1036\times10^{-5}$

Improving the block by considering additional equations arising from first and second derivative.
By letting

$$\frac{du(x)}{dx}=Z(x),\frac{du(x_0)}{dx}=Z_0$$

$$\frac{d^2u(x)}{dx^2}=Z'(x),\frac{d^2u(x_0)}{dx^2}=Z_0'$$

(19)

Evaluating the first and second derivative of the continuous scheme at $x = x_0$ using (11) gives t the value $\frac{-1}{2}$ .Thus the following derivative discrete schemes were obtained

$$hz_o + \frac{9}{2}y_n - 6y_{n+1/3} + \frac{3}{2}y_{n+2/3} = h^3\Big(\frac{2692147}{484989120}f_n + \frac{6174799}{146361600}f_{n+1/3} - \frac{761}{29568}f_{n+2/3}$$

$$+\frac{198359}{6928416}f_{n+1} - \frac{119981}{5702400}f_{n+4/3} + \frac{7523}{798336}f_{n+5/3} - \frac{3246301}{1616630400}f_{n+2} +$$

$$\frac{302161}{9699782400}f_{n+3} - \frac{4481}{4267904256}f_{n+4}\Big)$$

The order $P = 9$ and the Error constant $C_{12} = -2.0421 \times 10^{-8}$

$$h^2 z'_o - 9y_n + 18y_{n+1/3} - 9y_{n+2/3} = h^3\Big(-\frac{2314559}{24494400}f_n - \frac{14371477}{39916800}f_{n+1/3} + \frac{519971}{1814400}f_{n+2/3}$$

$$-\frac{772517}{2449440}f_{n+1} + \frac{843251}{3628800}f_{n+4/3} - \frac{189251}{1814400}f_{n+5/3} + \frac{121183}{5443200}f_{n+2}$$

$$-\frac{33937}{97977600}f_{n+3} + \frac{12601}{1077753600}f_{n+4}\Big)$$

(20)

The order $P = 9$ and the Error constant $C_{12} = -2.45836 \times 10^{-4}$

And imposing a continuity equation at $x = x_{n+4}$

$$hz_{n+4} - \frac{63}{2}y_n + 66y_{n+1/3} - \frac{69}{2}y_{n+2/3} = h^3\Big(-\frac{1488170869}{2424945600}f_n + \frac{2440841041}{439084800}f_{n+1/3}$$

$$-\frac{46073729}{2851200}f_{n+2/3} + \frac{8273785897}{242494560}f_{n+1} - \frac{1454116087}{39916800}f_{n+4/3} + \frac{484548391}{19958400}f_{n+5/3}$$

$$-\frac{8776922113}{1616630400}f_{n+2} + \frac{12895271941}{9699782400}f_{n+3} - \frac{5215046803}{106697606400}f_{n+4}\Big)$$

The order $P = 9$ and the Error constant $C_{12} = -7.71145 \times 10^{-5}$

$$h^2 z'_{n+4} - 9y_n + 18y_{n+1/3} - 9y_{n+2/3} = h^3\Big(-\frac{49841471}{24494400}f_n + \frac{60381217}{3628800}f_{n+1/3}$$

$$-\frac{103616221}{1814400}f_{n+2/3} + \frac{273689179}{2449440}f_{n+1} - \frac{464410381}{3628800}f_{n+4/3} + \frac{154335421}{1814400}f_{n+5/3}$$

$$-\frac{19670039}{777600}f_{n+2} + \frac{235609967}{97977600}f_{n+3} + \frac{23815499}{97977600}f_{n+4}\Big)$$

(21)

The order $P = 9$ and the Error constant $C_{12} = -2.45836 \times 10^{-4}$

Combining the discrete schemes in (18) and (20) forms the 4-step block method, which will be used simultaneously to generate solutions at both the steps and off-step points. And we explicitly obtain the initial conditions at x_{n+4}, $n = 0, 4, \ldots N-4$ using computed values of $u(x_{n+4}) = y_{n+4}$, $z(x_{n+4}) = z_{n+4}$, $z'(x_{n+4}) = z'_{n+4}$,over subintervals $\{x_0, x_4\}$, $\{x_4, x_8\}$, $\ldots$ $\{x_{N-4}, x_N\}$ that do not overlap.

Stability Analysis

The discrete and derivative schemes were put in matrix equation form and for easy analysis the result was normalized to obtain

$$
\begin{bmatrix}
1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1
\end{bmatrix}
\begin{bmatrix} y_{n+1/3}\\ y_{n+2/3}\\ y_{n+1}\\ y_{n+4/3}\\ y_{n+5/3}\\ y_{n+2}\\ y_{n+3}\\ y_{n+4} \end{bmatrix}
=
\begin{bmatrix}
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1
\end{bmatrix}
\begin{bmatrix} y_{n-7}\\ y_{n-6}\\ y_{n-5}\\ y_{n-4}\\ y_{n-3}\\ y_{n-2}\\ y_{n-1}\\ y_{n} \end{bmatrix}
+h^3
\begin{bmatrix}
\frac{1341139}{225815040} & \frac{-2637631}{359251200} & \frac{11607931}{1454967360} & \frac{-4236559}{718502400} & \frac{953311}{359251200} & \frac{-1100903}{1939956480} & \frac{73657}{8314099200} & \frac{-191783}{640185638400}\\
\frac{3203533}{61746300} & \frac{-130583}{2806650} & \frac{1159406}{22733865} & \frac{-21113}{561330} & \frac{47423}{2806650} & \frac{-273463}{75779550} & \frac{5111}{90935460} & \frac{-4747}{2500725150}\\
\frac{1679859}{10841600} & \frac{-6777}{70400} & \frac{799}{6336} & \frac{-18387}{197120} & \frac{20709}{492800} & \frac{-13291}{1478400} & \frac{83}{591360} & \frac{-463}{97574400}\\
\frac{973376}{3087315} & \frac{-194944}{1403325} & \frac{1152512}{4546773} & \frac{-34964}{200475} & \frac{110464}{1403325} & \frac{-127616}{7577955} & \frac{29888}{113669325} & \frac{-1588}{178623225}\\
\frac{168517625}{316141056} & \frac{-2512375}{14370048} & \frac{131479375}{290993472} & \frac{-7535875}{28740096} & \frac{1819375}{14370048} & \frac{-10488875}{387991296} & \frac{983375}{2327947776} & \frac{-365875}{25607425536}\\
\frac{13689}{16940} & \frac{-783}{3850} & \frac{2494}{3465} & \frac{-648}{1925} & \frac{783}{3850} & \frac{-13}{330} & \frac{43}{69300} & \frac{-4}{190575}\\
\frac{4074381}{2168320} & \frac{59049}{492800} & \frac{5751}{4928} & \frac{531441}{985600} & \frac{-59049}{492800} & \frac{46899}{98560} & \frac{10359}{985600} & \frac{-1863}{10841600}\\
\frac{88128}{21175} & \frac{-3456}{1925} & \frac{22016}{3465} & \frac{972}{385} & \frac{3456}{1925} & \frac{8576}{5775} & \frac{192}{385} & \frac{1444}{190575}
\end{bmatrix}
\begin{bmatrix} f_{n+1/3}\\ f_{n+2/3}\\ f_{n+1}\\ f_{n+4/3}\\ f_{n+5/3}\\ f_{n+2}\\ f_{n+3}\\ f_{n+4} \end{bmatrix}
$$

$$
+h^3
\begin{bmatrix}
0&0&0&0&0&0&0&\frac{49458977}{14549673600}\\
0&0&0&0&0&0&0&\frac{1966241}{113669325}\\
0&0&0&0&0&0&0&\frac{31039}{739200}\\
0&0&0&0&0&0&0&\frac{8808496}{113669325}\\
0&0&0&0&0&0&0&\frac{72043375}{581986944}\\
0&0&0&0&0&0&0&\frac{3134}{17325}\\
0&0&0&0&0&0&0&\frac{105597}{246400}\\
0&0&0&0&0&0&0&\frac{3952}{5775}
\end{bmatrix}
\begin{bmatrix} f_{n-7}\\ f_{n-6}\\ f_{n-5}\\ f_{n-4}\\ f_{n-3}\\ f_{n-2}\\ f_{n-1}\\ f_{n} \end{bmatrix}
\qquad (22)
$$

The first characteristics polynomial is given as

$$P(R) = \det\left(R \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right) \tag{23}$$

$= R^7(R-1) = 0$, which implies

$R_1 = 0, R_2 = 0, R_3 = 0, R_4 = 0, R_5 = 0, R_6 = 0, R_7 = 0$

and $R_8 = 1$ Therefore, the derived block method is zero stable since $|R| = 1$ is simple, and the magnitude of other roots $|R| = 0$. Also the method is consistent since the method as an order of $p > 1$. Hence the proposed method is convergent.

Region of Absolute Stability

The region of Absolute stability of the derived block method was plotted using idea of [6]

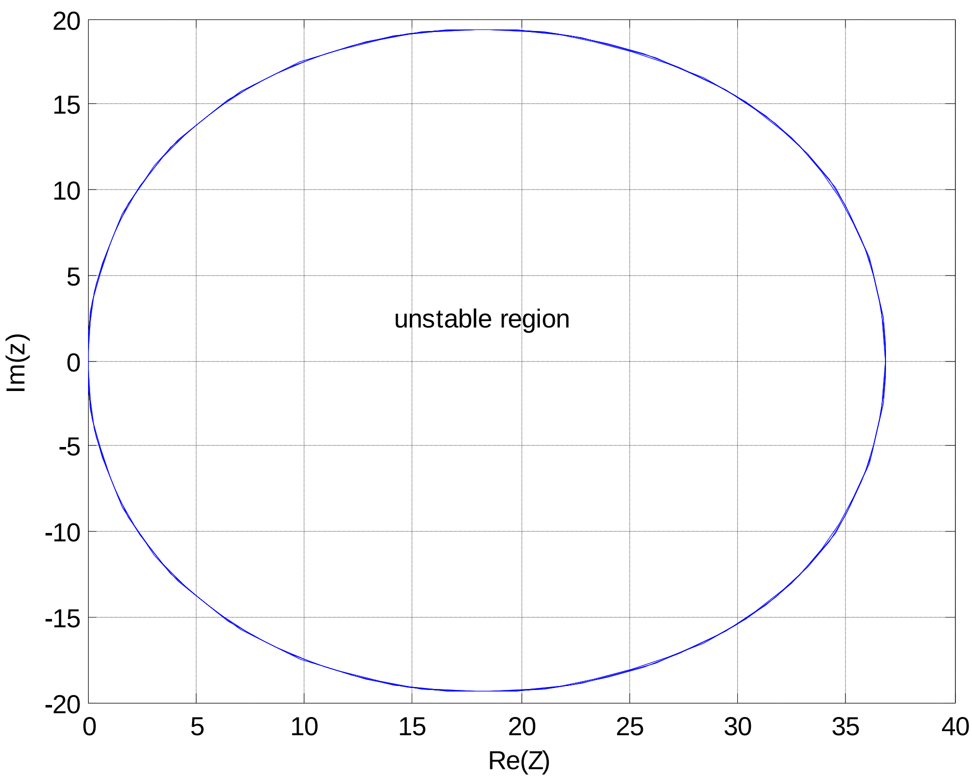

Fig. 1 Region of Absolute Stability for Case k = 4 with four Off- Step Points at Collocation and two Off- Step Points at Interpolation

From definition 7 the block is A-stable.

Numerical Experiment
Problem 1

$y''' = e^x$ (Non-Stiff)

$y(0) = 3 \ , y'(0) = 1 \ , y''(0) = 5$

Exact solution is given as $y(x) = 2 + 2x^2 + e^x$

Source:[11]

Problem 2

$y''' = -y$ (Stiff problem.)

$y(0) = 1 \ , y'(0) = -1 \ , y''(0) = 1 \ , \ 0 \le x \le 1 \ , \ h = 0.1$

Exact solution is given as $y(x) = e^{-x}$

Source: [8]

Problem 3

$y''' = 3\cos(x)$ (Non -Stiff problem)

$y(0) = 1 \ , y'(0) = 0 \ , y''(0) = 2 \ , h = 0.05$

Exact solution is given as $y(x) = x^2 + 3x - 3\sin(x) + 1$

Source:[11

TABLE I

Result of problem 1 from Four Steps with four Off- Step Points at Collocation and two Off- Step Points at Interpolation method

x	Exact solution	Computed results P=9	Error in Derived Method	Error in [4] p=9
0.1	3.12517091807565	3.12517091807565	0.0000E+00	0.000E+00
0.2	3.30140275816017	3.30140275816017	0.0000E+00	2.8422E−13
0.3	3.52985880757600	3.52985880757600	0.0000E+00	1.6729E−12
0.4	3.81182469764127	3.81182469764127	0.0000E+00	2.9983E−11
0.5	4.14872127070013	4.14872127070013	0.0000E+00	3.1673E−11
0.6	4.54211880039051	4.54211880039051	0.0000E+00	9.1899E−11
0.7	4.99375270747048	4.99375270747048	0.0000E+00	8.9531E−11
0.8	5.50554092849247	5.50554092849247	0.0000E+00	1.9168E−10
0.9	6.07960311115695	6.07960311115695	0.0000E+00	2.1110E−10
1.0	6.71828182845905	6.71828182845905	0.0000E+00	4.9398E−10

TABLE II

Result of problem 2 from four steps with four Off- Step Points at Collocation and two Off- Step Points at Interpolation method

x	**Exact solution**	**Computed results P=9**	**Error in Derived method**	**Error in [4] P=9**
1	0.90483741803595	0.90483741803596	1.0000E-14	0.0000E+00
0.2	0.81873075307798	0.81873075307798	0.0000E+00	2.7756E−14
0.3	0.74081822068172	0.74081822068172	0.0000E+00	1.5838E−12
0.4	0.67032004603564	0.67032004603564	0.0000E+00	2.7879E−11
0.5	0.60653065971263	0.60653065971263	0.0000E+00	2.9477E−11
0.6	0.54881163609403	0.54881163609403	0.0000E+00	8.5048E−11
0.7	0.49658530379141	0.49658530379141	0.0000E+00	8.0357E−11
0.8	0.44932896411722	0.44932896411722	0.0000E+00	1.6601E−10
0.9	0.40656965974060	0.40656965974060	0.0000E+00	1.1176E−10
1.0	0.36787944117144	0.36787944117145	1.0000E-14	1.4871E−10

TABLE III

Result of problem 3 from four steps with four Off- Step Poins at Collocation and two Off- Step Points at Interpolation method

x	**Exact solution**	**Computed result P=9**	**Error in Derived method**
0.05	1.00256249218797	1.00256249218797	0.0000E+00
0.1	1.01049975005952	1.01049975005952	0.0000E+00
0.15	1.02418560257920	1.02418560257920	0.0000E+00
0.20	1.04399200761482	1.04399200761482	0.0000E+00
0.25	1.07028812223643	1.07028812223643	0.0000E+00
.30	1.10343938001598	1.10343938001598	0.0000E+00
0.35	1.14380657763365	1.14380657763365	0.0000E+00
0.40	1.19174497307405	1.19174497307405	0.0000E+00
0.45	1.24760339766631	1.24760339766631	0.0000E+00
0.5	1.31172338418739	1.31172338418739	0.0000E+00

III. CONCLUSION

The introduction off-step points at both collocation and interpolation in developing block methods results in a better approximation and higher order, thus enabling the schemes to bypass the Dalhquist Barrier theorem.

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